k-SYMPLECTIC FORMALISM ON LIE ALGEBROIDS

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Abstract. In this paper we introduce a geometric description of Lagrangian and Hamiltonian classical field theories on Lie algebroids in the framework of k-symplectic geometry. We discuss the relation between Lagrangian and Hamiltonian descriptions through a convenient notion of Legendre transformation. The theory is a natural generalization of the standard one; in addition, other interesting examples are studied, in particular, systems with symmetry and Poisson sigma models.

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1. Introduction

The notion of Lie algebroid is a generalization of both the concept of a Lie algebra and the concept of an integrable distribution. The idea of using Lie algebroids in Mechanics is due to A. Weinstein [47]. He introduced a new geometric framework for the description of Lagrangian Mechanics. His formulation allows us to describe geometrically, in a unified

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way, different types of dynamical systems; such as those whose lagrangian systems whose phase spaces are Lie groups, Lie algebras, cartesian products of manifolds or quotient manifolds (as it happens, for instance, in the reduction theory, where the reduced phase spaces are not, in general, tangent or cotangent bundles). This approach was followed and completed by other authors in order to study different kinds of problems concerning mechanical systems (a good survey on this subject is [19]).

In this paper we will study an extension of mechanics on Lie algebroids to classical field theories. Classical field theories on Lie algebroids have already been studied in the literature. For instance, the multisymplectic formalism on Lie algebroids was presented in [32, 33]. In [45] a geometric framework for discrete field theories on Lie groupoids has been discussed.

The multisymplectic formalism was developed by Tulczyjews school in Warsaw (see, for instance, [17]), and independently by García and Pérez-Rendón [10, 11] and Goldschmidt and Sternberg [12]. This approach was revised, among others, by Martin [28, 29] and Gotay et al [13] and, more recently, by Cantrijn et al [9].

An alternative way to derive certain types of the field equations is to use the Günther (k-symplectic) formalism. The k-symplectic formalism is the generalization to field theories of the standard symplectic formalism in Mechanics, which is the geometric framework for describing autonomous dynamical systems. In this sense, the k-symplectic formalism is used to give a geometric description of certain kinds of field theories: in a local description, those theories whose Lagrangian and Hamiltonian do not depend on the base coordinates, denoted by $(t^1, \dots, t^k)$ (in many of the cases defining the space-time coordinates); that is, the k-symplectic formalism is only valid for Lagrangians $L(q^i, v^i_A)$ and Hamiltonians $H(q^i, p^A_i)$ that depend on the field coordinates q^i and on the partial derivatives of the field v^i_A, or the corresponding moment p^A_i. To treat with that general situation we need to extend the formalism using k-cosymplectic geometry, see [24, 25].

Günther's paper [14] gave a geometric Hamiltonian formalism for field theories. The crucial device is the introduction of a vector-valued generalization of a symplectic form, called a polysymplectic form. One of the advantages of this formalism is that one only needs the tangent and cotangent bundle of a manifold to develop it. In [36] Günther's formalism has been revised and clarified. It has been shown that the polysymplectic structures used by Günther to develop his formalism could be replaced by the k-symplectic structures defined independently by Awane [2, 3], L. K. Norris [34, 37, 38, 39, 40] and de Leon *et al.* [20, 22]. So this formalism is also called k-symplectic formalism (see also [21, 23]).

The purpose of this paper is to give a k-symplectic setting to first-order classical field theories on Lie algebroids. In the k-symplectic setting we will present a geometric description of Lagrangian and Hamiltonian classical field theories on Lie algebroids and we will find the relation between the solutions of both formalism when the Lagrangian is hyperregular.

The organization of the paper is as follows. In section 2 we recall some basic elements from the k-symplectic approach to first order classical field theories. In section 3 we recall some basic facts about Lie algebroids an the differential geometry associated to them. In this section we also describe a particular example of Lie algebroid, called the *prolongation of a Lie algebroid over a fibration.* This Lie algebroid will be necessary for the further developments. In section 4 the k-symplectic formalism is extended to the setting of Lie algebroids. The subsection 4.1 describe the Lagrangian approach and the subsection 4.2 describe the Hamiltonian approach. These formalism are developed in an analogous way to the standard k-symplectic Lagrangian and Hamiltonian formalism. We

finish this section defining the Legendre transformation on the context of Lie algebroids and we establish the equivalence between both formalism, Lagrangian and Hamiltonian, when the Lagrangian function is hyperregular. In section 5 we show some examples where the theory can be applied to the Poisson-Sigma model or first order field theories with symmetries.

All manifolds and maps are C^∞. Sum over crossed repeated indices is understood. Along this paper one k-tuple of elements will be denoted by a bold symbol.

2. Geometric preliminaries

In this section we recall some basic elements from the k-symplectic approach to classical field theories. The contents of this section can be found in [14, 36, 41].

2.1. The tangent bundle of k^1-velocities of a manifold.

Let $\tau_Q : TQ \to Q$ be the tangent bundle of Q, where Q is as n-dimensional differentiable manifold. Let us denote by T^1_kQ the Whitney sum $TQ\oplus \overset{k}{\ldots} \oplus TQ$ of k copies of TQ, with projection $\tau^k_Q : T^1_kQ \to Q$, $\tau^k_Q(v_{1q}, \ldots, v_{kq}) = q$, where $v_{Aq} \in T_qQ$, $A = 1, \ldots, k$.

T^1_kQ can be identified with the manifold $J^1_{\mathbf{0}}(\mathbb{R}^k, Q)$ of *k^1-velocities of* Q, that is, 1-jets of maps $\sigma : \mathbb{R}^k \to Q$ with source at $\mathbf{0} \in \mathbb{R}^k$, say

$$\begin{array}{ccc} J^1_{\mathbf{0}}(\mathbb{R}^k, Q) & \equiv & TQ\oplus \overset{k}{\ldots} \oplus TQ \\ j^1_{\mathbf{0},q}\sigma & \equiv & (v_{1q}, \ldots, v_{kq}) \end{array}$$

where $q = \sigma(\mathbf{0})$, and $v_{Aq} = \sigma_*(\mathbf{0})(\dfrac{\partial}{\partial t^A}\Big|_{\mathbf{0}})$. Here $(t^1, \ldots, t^k)$ denote the standard coordinates on $\mathbb{R}^k$. T^1_kQ is called *the tangent bundle of k^1-velocities of* Q (see [35]).

If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates (q^i, v^i), $1 \le i \le n$, on $TU = \tau_Q^{-1}(U)$ are expressed by

$$q^i(v_q) = q^i(q), \qquad v^i(v_q) = v_q(q^i)$$

and the induced local coordinates (q^i, v^i_A), $1 \le i \le n$, $1 \le A \le k$, on $T^1_kU = (\tau^k_Q)^{-1}(U)$ are given by

$$q^i(v_{1q}, \ldots, v_{kq}) = q^i(q), \qquad v^i_A(v_{1q}, \ldots, v_{kq}) = v_{Aq}(q^i)\,.$$

Let $f : M \to N$ be a differentiable map, then the induced map $T^1_kf : T^1_kM \to T^1_kN$ defined by $T^1_kf(j^1_{\mathbf{0}}\sigma) = j^1_{\mathbf{0}}(f \circ \sigma)$ is called the *canonical prolongation* of f. Observe that from its definition:

$$T^1_kf(v_{1q}, \ldots, v_{kq}) = (f_*(q)(v_{1q}), \ldots, f_*(q)(v_{kq})) \quad ,$$

where $v_{1q}, \ldots, v_{kq} \in T_qQ$, $q \in Q$.

2.2. k-vector fields and integral sections.

Let M be an arbitrary manifold.

Definition 2.1. *A section $\mathbf{X} : M \longrightarrow T^1_kM$ of the projection τ^k_M will be called a* k-vector field *on M.*

Since T^1_kM is the Whitney sum $TM\oplus \overset{k}{\ldots} \oplus TM$ of k copies of TM, we deduce that to give a k-vector field $\mathbf{X}$ is equivalent to give a family of k vector fields $X_1, \ldots, X_k$ on M obtained by projecting $\mathbf{X}$ on each factor. For this reason we will denote a k-vector field by $\mathbf{X} = (X_1, \ldots, X_k)$.

Definition 2.2. *An* integral section *of the k-vector field* $\mathbf{X} = (X_1, \dots, X_k)$, *passing through a point* $x \in M$, *is a map* $\psi \colon U_{\mathbf{0}} \subset \mathbb{R}^k \to M$, *defined on some neighborhood* $U_{\mathbf{0}}$ *of* $\mathbf{0} \in \mathbb{R}^k$, *such that*

$$\psi(\mathbf{0}) = x, \ \psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \right) = X_A(\psi(\mathbf{t})) \ , \quad \text{for every } \mathbf{t} \in U_{\mathbf{0}},\, 1 \leq A \leq k$$

or, equivalently, ψ *satisfies that* $\mathbf{X} \circ \psi = \psi^{(1)}$, *where* $\psi^{(1)}$ *is the first prolongation of* ψ *to* $T^1_k M$ *defined by*

$$\begin{array}{rccl} \psi^{(1)}: & U_0 \subset \mathbb{R}^k & \longrightarrow & T^1_k M \\ & \mathbf{t} & \longrightarrow & \psi^{(1)}(\mathbf{t}) = j^1_{\mathbf{0}} \psi_{\mathbf{t}} \equiv \left(\psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^1}\Big|_{\mathbf{t}} \right), \dots, \psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^k}\Big|_{\mathbf{t}} \right) \right) , \end{array}$$

where $\psi_{\mathbf{t}}(\mathbf{s}) = \psi(\mathbf{t} + \mathbf{s})$.

A k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ *on M is said to be* integrable *if there is an integral section passing through every point of M.*

Remark 2.3. In the k-symplectic formalism, the solutions of the field equations are described as the integral sections of some k-vector fields. Observe that, in the case $k = 1$, this definition coincides with the classical definition of integral curve of a vector field. $\diamond$

In a local coordinate system, if $\psi(\mathbf{t}) = (\psi^i(\mathbf{t}))$ then one has

$$\psi^{(1)}(\mathbf{t}) = \left(\psi^i(\mathbf{t}), \frac{\partial \psi^i}{\partial t^A}\Big|_{\mathbf{t}} \right), \qquad 1 \leq A \leq k\, ,\, 1 \leq i \leq n\, , \tag{2.1}$$

and ψ is an integral section of $(X_1, \dots, X_k)$ if and only if the following equations holds:

$$\frac{\partial \psi^i}{\partial t^A} = X^i_A \circ \psi \quad 1 \leq A \leq k,\ 1 \leq i \leq n\, , \tag{2.2}$$

being $X_A = X^i_A \dfrac{\partial}{\partial q^i}$.

2.3. **The cotangent bundle of k^1-covelocities of a manifold.** Let Q be a differentiable manifold of dimension n and $\pi_Q : T^*Q \to Q$ its cotangent bundle. Denote by $(T^1_k)^*Q = T^*Q \oplus \overset{k}{\dots} \oplus T^*Q$ the Whitney sum of T^*Q with itself k times, with projection map $\pi^k_Q \colon (T^1_k)^*Q \to Q$, $\pi^k_Q(\alpha_{1_q}, \dots, \alpha_{k_q}) = q$.

Observe that the manifold $(T^1_k)^*Q$ can be canonically identified with the vector bundle $J^1(Q, \mathbb{R}^k)_{\mathbf{0}}$ of k^1-covelocities of the manifold Q, the manifold of 1-jets of maps $\sigma \colon Q \to \mathbb{R}^k$ with target at $\mathbf{0} \in \mathbb{R}^k$ and projection map $\pi^k_Q \colon J^1(Q, \mathbb{R}^k)_{\mathbf{0}} \to Q$, $\pi^k_Q(j^1_{q,\mathbf{0}}\sigma) = q$; that is,

$$\begin{array}{ccc} J^1(Q, \mathbb{R}^k)_{\mathbf{0}} & \equiv & T^*Q \oplus \overset{k}{\dots} \oplus T^*Q \\ j^1_{q,\mathbf{0}}\sigma & \equiv & (d\sigma_1(q), \dots, d\sigma_k(q)) \end{array}$$

where $\sigma_A = pr_A \circ \sigma : Q \longrightarrow \mathbb{R}$ is the A-th component of σ, and $pr_A \colon \mathbb{R}^k \to \mathbb{R}$ are the canonical projections, $1 \leq A \leq k$. For this reason, $(T^1_k)^*Q$ is also called *the bundle of k^1-covelocities of the manifold Q*.

If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (q^i, p_i), $1 \leq i \leq n$, on $T^*U = (\pi_Q)^{-1}(U)$, are given by

$$q^i(\alpha_q) = q^i(q), \quad p_i(\alpha_q) = \alpha_q \left(\frac{\partial}{\partial q^i}\Big|_q \right)$$

and the induced local coordinates (q^i, p_i^A), $1 \leq i \leq n$, $1 \leq A \leq k$, on $(T_k^1)^*U = (\pi_Q^k)^{-1}(U)$ are

$$q^i(\alpha_{1_q}, \ldots, \alpha_{k_q}) = q^i(q), \qquad p_i^A(\alpha_{1_q}, \ldots, \alpha_{k_q}) = \alpha_{A_q}\left(\frac{\partial}{\partial q^i}\Big|_q\right).$$

We can endow $(T_k^1)^*Q$ with a k-symplectic structure given by the family $(\omega^1, \ldots, \omega^k; V = \ker T\pi_Q^k)$ where each ω^A is the 2-form given by

$$\omega^A = (\pi_Q^{k,A})^*\omega_Q, \quad 1 \leq A \leq k\,,$$

being $\pi_Q^{k,A}\colon (T_k^1)^*Q \to T^*Q$ the canonical projection onto the A^{th}-copy T^*Q of $(T_k^1)^*Q$ and ω_Q is the canonical sympletic form on T^*Q. Therefore, in local coordinates, we have $\omega^A = dq^i \wedge dp_i^A$. (See [2, 3, 36, 41])

3. Lie algebroids

In this section we present some basic facts on Lie algebroids, including results from the associated differential calculus and Lie algebroids morphisms, that will be necessary for the further developments. We refer the reader to [4, 15, 26, 27] for details about Lie groupoids, Lie algebroids and their role in differential geometry.

3.1. **Lie algebroid: definition.** Let E be a vector bundle of rank m over a manifold Q of dimension n and $\tau : E \to Q$ be the vector bundle projection. Denoted by $\mathrm{Sec}(E)$ the $C^\infty(Q)$-module of sections of $\tau : E \to Q$. A *Lie algebroid structure* $([\![\cdot,\cdot]\!]_E, \rho_E)$ on E is a Lie bracket $[\![\cdot,\cdot]\!]_E : \mathrm{Sec}(E) \times \mathrm{Sec}(E) \to \mathrm{Sec}(E)$ on the space $\mathrm{Sec}(E)$, together with a bundle map $\rho_E : E \to TQ$, called *the anchor map*, such that if we also denote by $\rho_E : \mathrm{Sec}(E) \to \mathfrak{X}(Q)$ the homomorphism of the $C^\infty(Q)$-modules induced by the anchor map then it is satisfied the following *compatibility condition*

$$[\![\sigma_1, f\sigma_2]\!]_E = f[\![\sigma_1, \sigma_2]\!]_E + (\rho_E(\sigma_1)f)\sigma_2\,.$$

Here f is a smooth function on Q, σ_1, σ_2 are sections of E and we have denoted by $\rho_E(\sigma_1)$ the vector field on Q given by $\rho_E(\sigma_1)(q) = \rho_E(\sigma_1(q))$. The triple $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ is called a *Lie algebroid over* Q. From the compatibility condition and the Jacobi identity, it follows that the anchor map $\rho_E : \mathrm{Sec}(E) \to \mathfrak{X}(Q)$ is a homomorphism between the Lie algebras $(\mathrm{Sec}(E), [\![\cdot,\cdot]\!]_E)$ and $(\mathfrak{X}(Q), [\cdot,\cdot])$.

Some examples of Lie algebroids over Q are:

(i) **Real Lie algebras of finite dimension**. Let $\mathfrak{g}$ be a real Lie algebra of finite dimension. Then, it is clear that $\mathfrak{g}$ is a Lie algebroid over a single point.

(ii) **The tangent bundle.** Let TQ be the tangent bundle of a manifold Q. Then, the triple $(TQ, [\cdot,\cdot], id_{TQ})$ is a Lie algebroid over Q, where $id_{TQ} : TQ \to TQ$ is the identity map.

(iii) Another interesting example of a Lie algebroid may be constructed as follows. Let $\pi : P \to Q$ be a principal bundle with structural group G. Denote by $\Phi : G \times P \to P$ the free action of G on P and by $T\Phi : G \times TP \to TP$ the tangent action of G on TP. Then, one may consider the quotient vector bundle $\tau_{P|G} : TP/G \to Q = P/G$ and the sections of this vector bundle may be identified with the vector fields on P which are invariant under the action Φ. Using that every G-invariant vector field on P is π-projectable and the fact that the standard Lie bracket on vector fields is closed with respect to G-invariant vector fields, we can induce a Lie algebroid structure on TP/G. The resultant Lie algebroid is called **the Atiyah (gauge) algebroid associated with the principal bundle** $\pi : P \to Q$ (see [19, 26]).

Along this paper, the Lie algebroid will play the role of a substitute of the tangent bundle of Q. In this way, one regards an element e of E as a generalized velocity, and the actual velocity v is obtained when we apply the anchor map to e, i.e. $v = \rho_E(e)$.

Let $(q^i)_{i=1}^n$ be local coordinates on Q and $\{e_\alpha\}_{1\leq\alpha\leq m}$ be a local basis of sections of τ. Given $e \in E$ such that $\tau(e) = q$, we can write $e = y^\alpha(e)e_\alpha(q) \in E_q$, thus the coordinates of e are $(q^i(e), y^\alpha(e))$. Therefore, each section σ is locally given by $\sigma\big|_U = y^\alpha e_\alpha$.

In local form, the Lie algebroid structure is determined by a set of local functions ρ^i_α, $\mathcal{C}^\gamma_{\alpha\beta}$ on Q. They are determined by the relations

$$\rho_E(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}, \quad [\![e_\alpha, e_\beta]\!]_E = \mathcal{C}^\gamma_{\alpha\,\beta} e_\gamma \,. \tag{3.1}$$

The functions ρ^i_α and $\mathcal{C}^\gamma_{\alpha\beta}$ are called the *structure functions* of the Lie algebroid in the above coordinate system. They satisfy the following relations (as a consequence of the compatibility condition and Jacobi's identity):

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left(\rho^i_\alpha \frac{\partial \mathcal{C}^\nu_{\beta\gamma}}{\partial q^i} + \mathcal{C}^\nu_{\alpha\mu}\mathcal{C}^\mu_{\beta\gamma} \right) = 0 \quad , \quad \rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial q^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^i_\gamma \mathcal{C}^\gamma_{\alpha\beta} \,, \tag{3.2}$$

which are usually called *the structure equations* of the Lie algebroid E.

3.2. **Exterior differential.** The structure of Lie algebroid on E allows us to define *the exterior differential of* E, $d^E : \mathrm{Sec}(\bigwedge^l E^*) \to \mathrm{Sec}(\bigwedge^{l+1} E^*)$, as follows

$$\begin{aligned} d^E\mu(\sigma_1, \dots, \sigma_{l+1}) &= \sum_{i=1}^{l+1} (-1)^{i+1} \rho_E(\sigma_i)\mu(\sigma_1, \dots, \widehat{\sigma_i}, \dots, \sigma_{l+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \mu([\sigma_i, \sigma_j]_E, \sigma_1, \dots, \widehat{\sigma_i}, \dots, \widehat{\sigma_j}, \dots \sigma_{l+1}) \,, \end{aligned} \tag{3.3}$$

for $\mu \in \mathrm{Sec}(\bigwedge^l E^*)$ and $\sigma_1, \dots, \sigma_{l+1} \in \mathrm{Sec}(E)$. It follows that d^E is a cohomology operator, that is, $(d^E)^2 = 0$.

In particular, if $f : Q \to \mathbb{R}$ is a real smooth function then $d^E f(\sigma) = \rho_E(\sigma) f$, for $\sigma \in \mathrm{Sec}(E)$. Locally, the exterior differential is determined by

$$d^E q^i = \rho^i_\alpha e^\alpha \quad \text{and} \quad d^E e^\gamma = -\frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta \,,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$.

The usual Cartan calculus extends to the case of Lie algebroids: for every section σ of E we have a derivation $\imath_\sigma$ (contraction) of degree -1 and a derivation $\mathcal{L}_\sigma = \imath_\sigma \circ d + d \circ \imath_\sigma$ (Lie derivative) of degree 0 (for more details, see [26, 27]).

3.3. **Morphisms.** Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ and $(E', [\![\cdot,\cdot]\!]_{E'}, \rho_{E'})$ be two Lie algebroids over Q and Q' respectively, and suppose that $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a vector bundle map, that is $\overline{\Phi} : E \to E'$ is a fiberwise linear map over $\underline{\Phi} : Q \to Q'$. The pair $(\overline{\Phi}, \underline{\Phi})$ is said to be a *Lie algebroid morphism* if

$$d^E(\Phi^*\sigma') = \Phi^*(d^{E'}\sigma') \,, \quad \text{for all } \sigma' \in Sec(\textstyle\bigwedge^l (E')^*) \text{ and for all } l. \tag{3.4}$$

Here $\Phi^*\sigma'$ is the section of the vector bundle $\bigwedge^l E^* \to Q$ defined (for $l > 0$) by

$$(\Phi^*\sigma')_q(e_1, \dots, e_l) = \sigma'_{\underline{\Phi}(q)}(\overline{\Phi}(e_1), \dots, \overline{\Phi}(e_l)) \,, \tag{3.5}$$

for $q \in Q$ and $e_1, \ldots, e_l \in E_q$. In particular when $Q = Q'$ and $\underline{\Phi} = id_Q$ then (3.4) holds if and only if

$$[\![\overline{\Phi} \circ \sigma_1, \overline{\Phi} \circ \sigma_2]\!]_{E'} = \overline{\Phi}[\![\sigma_1, \sigma_2]\!]_E, \quad \rho_{E'}(\overline{\Phi} \circ \sigma) = \rho_E(\sigma), \quad \text{for } \sigma, \sigma_1, \sigma_2 \in Sec(E)\,.$$

Let (q^i) be a local coordinate system on Q and $(\bar{q}^{\bar{i}})$ a local coordinate system on Q'. Let $\{e_\alpha\}$ and $\{\bar{e}_{\bar{\alpha}}\}$ be a local basis of section of E and E', respectively, and $\{e^\alpha\}$ and $\{\bar{e}^{\bar{\alpha}}\}$ their dual basis, respectively. The vector bundle map Φ is determined by the relations $\Phi^*\bar{q}^{\bar{i}} = \phi^{\bar{i}}(q)$ and $\Phi^*\bar{e}^{\bar{\alpha}} = \phi^{\bar{\alpha}}_\beta e^\beta$ for certain local functions $\phi^{\bar{i}}$ and $\phi^{\bar{\alpha}}_\beta$ on Q. In this coordinate system $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a Lie algebroids morphism if and only if

$$(\rho_E)^j_\alpha \frac{\partial \phi^{\bar{i}}}{\partial q^j} = (\rho_{E'})^{\bar{i}}_{\bar{\beta}} \phi^{\bar{\beta}}_\alpha \quad , \quad \phi^{\bar{\beta}}_\gamma \mathcal{C}^\gamma_{\alpha\delta} = \left((\rho_E)^i_\alpha \frac{\partial \phi^{\bar{\beta}}_\delta}{\partial q^i} - (\rho_E)^i_\delta \frac{\partial \phi^{\bar{\beta}}_\alpha}{\partial q^i} \right) + \bar{\mathcal{C}}^{\bar{\beta}}_{\bar{\theta}\bar{\sigma}} \phi^{\bar{\theta}}_\alpha \phi^{\bar{\sigma}}_\delta\,. \tag{3.6}$$

In these expressions $(\rho_E)^i_\alpha, \mathcal{C}^\alpha_{\beta\gamma}$ are the structure functions on E, and $(\rho_{E'})^{\bar{i}}_{\bar{\alpha}}, \bar{\mathcal{C}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ are the structure functions on E'.

3.4. **The prolongation of a Lie algebroid over a fibration.** (See [8, 15, 19, 30]). In this subsection we describe a particular example of Lie algebroid which will be necessary for the further developments.

Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ be a Lie algebroid over a manifold Q and $\pi : P \to Q$ be a fibration. We consider the subset of $E \times TP$

$$\mathcal{T}^E_p P = \{(e, v_p) \in E_q \times T_p P \,|\, \rho_E(e) = T_p\pi(v_p)\}\,,$$

where $T\pi : TP \to TQ$ is the tangent map to π , $p \in P$ and $\pi(p) = q$.

$$\mathcal{T}^E P = \bigcup_{p \in P} \mathcal{T}^E_p P$$

is a vector bundle over P with projection $\widetilde{\tau}_P : \mathcal{T}^E P \to P$ given by

$$\widetilde{\tau}_P(e, v_p) = \tau_P(v_p) = p\,,$$

being $\tau_P : TP \to P$ the canonical projection.

Next, we will see that it is possible to induce a Lie algebroid structure on $\widetilde{\tau}_P : \mathcal{T}^E P \to P$. The anchor map ρ^π is given as follows: $\rho^\pi : \mathcal{T}^E P \to TP$, $\rho^\pi(e, v_p) = v_p$.

In order to introduce a Lie bracket on $\mathrm{Sec}(\mathcal{T}^E P)$, the set of sections of $\widetilde{\tau}_P$, we first consider a local basis of $\mathrm{Sec}(\mathcal{T}^E P)$.

Given local coordinates (q^i, u^ℓ) on P and a local basis $\{e_\alpha\}$ of sections of E, we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$ of sections of $\widetilde{\tau}_P : \mathcal{T}^E P \to P$ by

$$\mathcal{X}_\alpha(p) = (e_\alpha(\pi(p)); \rho^i_\alpha(\pi(p)) \frac{\partial}{\partial q^i}\Big|_p) \quad \text{and} \quad \mathcal{V}_\ell(p) = (0_{\pi(p)}; \frac{\partial}{\partial u^\ell}\Big|_p)\,. \tag{3.7}$$

If $z = (e, v_p)$ is an element of $\mathcal{T}^E P$, with $e = z^\alpha e_\alpha$, then v_p is of the form $v_p = \rho^i_\alpha z^\alpha \frac{\partial}{\partial q^i}\Big|_p + v^\ell \frac{\partial}{\partial u^\ell}\Big|_p$, and we can write

$$z = z^\alpha \mathcal{X}_\alpha(p) + v^\ell \mathcal{V}_\ell(p)\,.$$

The anchor map ρ^π applied to a section Z of $\mathcal{T}^E P$ with local expression $Z = Z^\alpha \mathcal{X}_\alpha + V^\ell \mathcal{V}_\ell$ is the vector field on P whose coordinate expression is

$$\rho^\pi(Z) = \rho^i_\alpha Z^\alpha \frac{\partial}{\partial q^i} + V^\ell \frac{\partial}{\partial u^\ell} \in \mathfrak{X}(P)\,. \tag{3.8}$$

Now, we will introduce a Lie bracket structure on the space of sections of $\mathcal{T}^E P$. For that, we say that a section Z of $\mathcal{T}^E P$ is *projectable* if there exists a section σ of $\tau\colon E \to Q$ and a vector field $X \in \mathfrak{X}(P)$, which is π-projectable to the vector field $\rho(\sigma)$ and such that $Z(p) = (\sigma(\pi(p)), X(p))$, for all $p \in P$. For such a projectable section Z, we will use the following notation $Z \equiv (\sigma, X)$. It is easy to prove that one may choose a local basis of projectable sections of the space $\mathrm{Sec}(\mathcal{T}^E P)$.

The Lie bracket of two projectable sections $Z = (\sigma, X)$ and $Z' = (\sigma', X')$ is then given by

$$\llbracket Z, Z'\rrbracket^\pi(p) = (\llbracket\sigma, \sigma'\rrbracket_E(q), [X, X'](p)), \quad p \in P,\ q = \pi(p)\,. \tag{3.9}$$

Since any section of $\mathcal{T}^E P$ can be locally written as a linear combination of projectable sections, the definition of the Lie bracket for arbitrary sections of $\mathcal{T}^E P$ follows. In particular, the Lie bracket of the elements of the local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$ of $\mathrm{Sec}(\mathcal{T}^E P)$ is characterized by the following expressions

$$\llbracket\mathcal{X}_\alpha, \mathcal{X}_\beta\rrbracket^\pi = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \quad \llbracket\mathcal{X}_\alpha, \mathcal{V}_\ell\rrbracket^\pi = 0 \quad \llbracket\mathcal{V}_\ell, \mathcal{V}_\varphi\rrbracket^\pi = 0\,, \tag{3.10}$$

and, therefore, the exterior differential is determined by

$$\begin{array}{rclcrcl} d^{\mathcal{T}^E P} q^i & = & \rho^i_\alpha \mathcal{X}^\alpha\,, & \qquad & d^{\mathcal{T}^E P} u^\ell & = & \mathcal{V}^\ell \\ d^{\mathcal{T}^E P} \mathcal{X}^\gamma & = & -\dfrac{1}{2}\mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta\,, & & d^{\mathcal{T}^E P}\mathcal{V}^\ell & = & 0 \end{array} \tag{3.11}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^\ell\}$ is the dual basis to $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$.

The Lie algebroid $\mathcal{T}^E P$ is called the *prolongation of the Lie algebroid E over the fibration* $\pi\colon P \to Q$. This Lie algebroid is very important in the k-symplectic formalism on Lie algebroids as we will see in the following section.

4. Classical Field Theories on Lie algebroids: k-symplectic approach

In this section, the k-symplectic formalism for first order classical field theories (see [14, 36, 41]) is extended to the setting of Lie algebroids. Thinking on a Lie algebroid E as a generalization of the tangent bundle of Q, we define the analog of the concept of field solution to the field equations and we study the analog of the geometric structures of the standard k-symplectic formalism.

In this section we will develop the Lagrangian and Hamiltonian k-symplectic formalism on Lie algebroids (see subsections 4.1 and 4.2). Moreover, we also describe the standard Lagrangian and Hamiltonian k-symplectic formalism as a particular case of the formalism developed here.

Along this section we consider a Lie algebroid $(E, \llbracket\cdot,\cdot\rrbracket_E, \rho_E)$ on the manifold Q. We note this Lie algebroid by E.

4.1. **Lagrangian formalism.**

4.1.1. *The manifold* $\stackrel{k}{\oplus} E$. The standard k-symplectic Lagrangian formalism is developed on the bundle of k^1-velocities of Q, T^1_kQ, that is the Whitney sum of k copies of TQ. Since we are thinking on a Lie algebroid E as a substitute of the tangent bundle, its natural to think that in this situation, the analog of the bundle of k^1-velocities T^1_kQ is the Whitney sum of k copies of the algebroid E.

We denote by

$$\stackrel{k}{\oplus} E = E \oplus \stackrel{k}{\ldots} \oplus E,$$

the Whitney sum of k copies of the vector bundle E, with projection map

$$\widetilde{\tau} : \overset{k}{\oplus} E \to Q,$$

given by $\widetilde{\tau}(e_{1_q}, \dots, e_{k_q}) = q$.

If (q^i, y^α) are local coordinates on $\tau^{-1}(U) \subseteq E$, then the induced local coordinates (q^i, y^α_A) on $\widetilde{\tau}^{-1}(U) \subseteq \overset{k}{\oplus} E$ are given by

$$q^i(e_{1_q}, \dots, e_{k_q}) = q^i(q)\,, \quad y^\alpha_A(e_{1_q}, \dots, e_{k_q}) = y^\alpha(e_{A_q})\,.$$

Remark 4.1. Consider the standard case where $E = TQ$, $\rho_{TQ} = id_{TQ}$. If we fix local coordinates (q^i) on Q, then we have the natural basis of $\mathrm{Sec}(TQ) = \mathfrak{X}(Q)$ given by $\left\{\dfrac{\partial}{\partial q^i}\right\}_{i=1}^n$. For this basis of section, obviously we have that $\mathcal{C}^\gamma_{\alpha\beta} = 0$; moreover, the set $\mathrm{Sec}\,(\overset{k}{\oplus} TQ) = \mathrm{Sec}\,(T^1_k Q)$ is the set $\mathfrak{X}^k(Q)$ of k-vectors fields on Q. ◇

4.1.2. *The Lagragian prolongation.* For the description of the Lagrangian k-symplectic formalism on lie algebroids we consider the prolongation of a Lie algebroid E over the fibration $\widetilde{\tau}\colon \overset{k}{\oplus} E \to Q$, that is, (see Section 3.4),

$$\mathcal{T}^E(\overset{k}{\oplus} E) = \{(e_q, v_{\mathbf{b}_q}) \in E \times T(\overset{k}{\oplus} E) /\ \rho_E(e_q) = T\widetilde{\tau}(v_{\mathbf{b}_q})\}\,, \tag{4.1}$$

where $\mathbf{b}_q \in \overset{k}{\oplus} E_q$. Taking into account the description of the prolongation $\mathcal{T}^E P$ (see for instance, [8, 19, 30]) on the particular case $P = E \oplus \overset{k}{\dots} \oplus E$ we obtain:

(i) $\mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$ is a Lie algebroid over $\overset{k}{\oplus} E$ with projection

$$\widetilde{\tau}_{\overset{k}{\oplus}E} \colon \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \longrightarrow \overset{k}{\oplus} E$$

and Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket^{\widetilde{\tau}}, \rho^{\widetilde{\tau}})$ where the anchor map

$$\rho^{\widetilde{\tau}} = E \times_{TQ} T(\overset{k}{\oplus} E) \colon \mathcal{T}^E(\overset{k}{\oplus} E) \to T(\overset{k}{\oplus} E)$$

is the canonical projection over the second factor.

We will refer to this particular Lie algebroid as the *Lagrangian prolongation.*

(ii) If (q^i, y^α_A) denotes a local coordinate system of $\overset{k}{\oplus} E$ then the induced local coordinate system on $\mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$ is given by

$$(q^i, y^\alpha_A, z^\alpha, w^\alpha_A)_{1\leq i\leq n,\ 1\leq A\leq k,\ 1\leq\alpha\leq m}$$

where

$$\begin{array}{rclcrcl} q^i(e_q, v_{\mathbf{b}_q}) &=& q^i(q)\,, & & y^\alpha_A(e_q, v_{\mathbf{b}_q}) &=& y^\alpha_A(\mathbf{b}_q)\,, \\ z^\alpha(e_q, v_{\mathbf{b}_q}) &=& y^\alpha(e_q)\,, & & w^\alpha_A(e_q, v_{\mathbf{b}_q}) &=& v_{\mathbf{b}_q}(y^\alpha_A)\,. \end{array} \tag{4.2}$$

(iii) The set $\{\mathcal{X}_\alpha, \mathcal{V}^A_\alpha\}$ given by

$$\begin{array}{rcrcrcl} \mathcal{X}_\alpha\colon & \overset{k}{\oplus} E & \to & \mathcal{T}^E(\overset{k}{\oplus} E) & \equiv & E \times_{TQ} T(\overset{k}{\oplus} E) \\ & \mathbf{b}_q & \mapsto & \mathcal{X}_\alpha(\mathbf{b}_q) & = & (e_\alpha(q); \rho^i_\alpha(q)\dfrac{\partial}{\partial q^i}\Big|_{\mathbf{b}_q}) \\ & & & & & \\ \mathcal{V}^A_\alpha\colon & \overset{k}{\oplus} E & \to & \mathcal{T}^E(\overset{k}{\oplus} E) & \equiv & E \times_{TQ} T(\overset{k}{\oplus} E) \\ & \mathbf{b}_q & \mapsto & \mathcal{V}^A_\alpha(\mathbf{b}_q) & = & (0_q; \dfrac{\partial}{\partial y^\alpha_A}\Big|_{\mathbf{b}_q})\,, \end{array} \tag{4.3}$$

is a local basis of $Sec(\mathcal{T}^E(\overset{k}{\oplus} E))$ the set of sections of $\widetilde{\tau}_{\overset{k}{\oplus}E}$. (See (3.7)).

(iv) The anchor map $\rho^{\widetilde{\tau}}\colon \mathcal{T}^E(\overset{k}{\oplus} E) \to T(\overset{k}{\oplus} E)$ allows us to associate a vector field to each section $\xi\colon \overset{k}{\oplus} E \to \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$ of $\widetilde{\tau}_{\overset{k}{\oplus}E}$.

Locally, if ξ writes as follows:

$$\xi = \xi^\alpha \mathcal{X}_\alpha + \xi^\alpha_A \mathcal{V}^A_\alpha \in Sec(\mathcal{T}^E(\overset{k}{\oplus} E))$$

then the associated vector field has the following local expression, see (3.8),

$$\rho^{\widetilde{\tau}}(\xi) = \rho^i_\alpha \xi^\alpha \frac{\partial}{\partial q^i} + \xi^\alpha_A \frac{\partial}{\partial y^\alpha_A} \in \mathfrak{X}(\overset{k}{\oplus} E)\,. \tag{4.4}$$

(v) The Lie bracket of two sections of $\widetilde{\tau}_{\overset{k}{\oplus}E}$ is characterized by the following expressions (see (3.10)):

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\widetilde{\tau}} = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \quad [\![\mathcal{X}_\alpha, \mathcal{V}^B_\beta]\!]^{\widetilde{\tau}} = 0 \quad [\![\mathcal{V}^A_\alpha, \mathcal{V}^B_\beta]\!]^{\widetilde{\tau}} = 0\,, \tag{4.5}$$

(vi) If $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha_A\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}^A_\alpha\}$, then the exterior differential is locally given by, (see (3.11)),

$$\begin{aligned} d^{\mathcal{T}^E(\overset{k}{\oplus} E)} f &= \rho^i_\alpha \frac{\partial f}{\partial q^i}\mathcal{X}^\alpha + \frac{\partial f}{\partial y^\alpha_A}\mathcal{V}^\alpha_A\,, \quad \text{for all } f \in \mathcal{C}^\infty(\overset{k}{\oplus} E)\\ d^{\mathcal{T}^E(\overset{k}{\oplus} E)}\mathcal{X}^\gamma &= -\frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha\wedge\mathcal{X}^\beta, \quad d^{\mathcal{T}^E(\overset{k}{\oplus} E)}\mathcal{V}^\gamma_A = 0\,. \end{aligned} \tag{4.6}$$

Remark 4.2. In the particular case $E = TQ$ the manifold $\mathcal{T}^E(\overset{k}{\oplus} E)$ turns into $T(T^1_kQ)$. In fact, in this case we consider the prolongation of TQ over $\tau^k_Q : T^1_kQ \to Q$. Thus from (4.1) we obtain

$$\begin{aligned} & \mathcal{T}^{TQ}(\overset{k}{\oplus} TQ) = \mathcal{T}^{TQ}(T^1_kQ)\\ = & \{(u_q, v_{\mathbf{w}_q}) \in TQ\times T(T^1_kQ)/ u_q = T(\tau^k_Q)(v_{\mathbf{w}_q})\}\\ = & \{(T(\tau^k_Q)(v_{\mathbf{w}_q}), v_{\mathbf{w}_q}) \in TQ\times T(T^1_kQ)/\ \mathbf{w}_q\in T^1_kQ\}\\ \equiv & \{v_{\mathbf{w}_q}\in T(T^1_kQ)/\ \mathbf{w}_q \in T^1_kQ\} \equiv T(T^1_kQ) \end{aligned} \tag{4.7}$$

◇

4.1.3. *The Liouville sections and the vertical endomorphism.* On $\mathcal{T}^E(\overset{k}{\oplus} E)$ we are going to define two families of canonical objects: *Liouville sections* and *vertical endomorphism* which corresponds with *the Liouville vector fields* and *the k-tangent structure* on T^1_kQ, when we consider the particular case $E = TQ$. (See [14, 36, 41]).

Vertical A-lift. (See, for instance [8]). An element $(e_q, v_{\mathbf{b}_q})$ of $\mathcal{T}^E(\overset{k}{\oplus} E) \equiv E\times_{TQ} T(\overset{k}{\oplus} E)$ is said to be *vertical* if

$$\widetilde{\tau}_1(e_q, v_{\mathbf{b}_q}) = 0_q \in E\,, \tag{4.8}$$

where

$$\begin{aligned} \widetilde{\tau}_1 :\ \ \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E\times_{TQ} T(\overset{k}{\oplus} E) &\to E,\\ (e_q, v_{\mathbf{b}_q}) &\mapsto \widetilde{\tau}_1(e_q, v_{\mathbf{b}_q}) = e_q \end{aligned}$$

is the projection on the first factor E of $\mathcal{T}^E(\overset{k}{\oplus} E)$.

The above definition implies that the vertical elements of $\mathcal{T}^E(\overset{k}{\oplus} E)$ are of the form

$$(0_q, v_{\mathbf{b}_q}) \in \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$$

where $v_{\mathbf{b}_q} \in T(\overset{k}{\oplus} E)$ and $\mathbf{b}_q \in \overset{k}{\oplus} E$.

Now, taking into account the definition (4.1), which determines the elements of $\mathcal{T}^E(\overset{k}{\oplus} E)$, the condition (4.8) means that

$$0_q = T_{\mathbf{b}_q}\widetilde{\tau}(v_{\mathbf{b}_q}),$$

that is, the tangent vector $v_{\mathbf{b}_q}$ is $\widetilde{\tau}$-vertical.

In a local coordinate system (q^i, y^α_A) on $\overset{k}{\oplus} E$, if $(e_q, v_{\mathbf{b}_q}) \in \mathcal{T}^E(\overset{k}{\oplus} E)$ is vertical then $e_q = 0_q$ and

$$v_{\mathbf{b}_q} = u^\alpha_A \frac{\partial}{\partial y^\alpha_A}\Big|_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E).$$

Definition 4.3. *For each $A = 1, \dots, k$ we call the vertical A^{th}-lifting map to the mapping*

$$\begin{array}{rcl} \xi^{V_A} : E \times_Q (\overset{k}{\oplus} E) & \longrightarrow & \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \\ (e_q, \mathbf{b}_q) & \longmapsto & \xi^{V_A}(e_q, \mathbf{b}_q) = \Big(0_q, (e_q)^{V_A}_{\mathbf{b}_q}\Big) \end{array} \tag{4.9}$$

where $e_q \in E$, $\mathbf{b}_q = (b_{1q}, \dots, b_{kq}) \in \overset{k}{\oplus} E$ and the vector $(e_q)^{V_A}_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$ is given by

$$(e_q)^{V_A}_{\mathbf{b}_q} f = \frac{d}{ds}\Big|_{s=0} f(b_{1_q}, \dots, b_{A_q} + s e_q, \dots, b_{k_q}), \quad 1 \leq A \leq k, \tag{4.10}$$

for an arbitrary function $f \in \mathcal{C}^\infty(\overset{k}{\oplus} E)$.

From (4.10) we deduce that the local expression of $(e_q)^{V_A}_{\mathbf{b}_q}$ is the following:

$$(e_q)^{V_A}_{\mathbf{b}_q} = y^\alpha(e_q) \frac{\partial}{\partial y^\alpha_A}\Big|_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E), \quad 1 \leq A \leq k. \tag{4.11}$$

On (4.11) let us observe that the vector $(e_q)^{V_A}_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$ is $\widetilde{\tau}$-vertical. Then $\xi^{V_A}(e_q, \mathbf{b}_q)$ is a vertical element of $\mathcal{T}^E(\overset{k}{\oplus} E)$.

From (4.3), (4.9) and (4.11) we obtain that ξ^{V_A} has the following local expression:

$$\xi^{V_A}(e_q, \mathbf{b}_q) \; = \; (0_q, y^\alpha(e_q) \frac{\partial}{\partial y^\alpha_A}\Big|_{\mathbf{b}_q}) = y^\alpha(e_q) \mathcal{V}^A_\alpha(\mathbf{b}_q), \quad 1 \leq A \leq k. \tag{4.12}$$

Remark 4.4.

(i) In the standard case, that is, when $E = TQ$ and $\rho_{TQ} = id_{TQ}$, we have that given $e_q \in T_qQ$ and $\mathbf{v}_q = (v_{1q}, \dots, v_{kq}) \in T^1_kQ$ one has

$$(e_q)^{V_A}_{\mathbf{v}_q}(f) = \frac{d}{ds}\Big|_{s=0} f(v_{1_q}, \dots, v_{A_q} + s e_q, \dots, v_{k_q}), \quad 1 \leq A \leq k,$$

that is, the A^{th}- vertical lift to T^1_kQ of the tangent vector $e_q \in T_qQ$ (see for instance, [14, 36, 41]).

(ii) In the particular case $k = 1$ we obtain that $\xi^{V_1} \equiv \xi^V : E \times_Q E \to \mathcal{T}^E E$ is the *vertical lifting map* introduced by E. Martínez in [30].

◇

The Liouville sections. The A^{th}-*Liouville section* $\widetilde{\Delta}_A$ is the section of $\widetilde{\tau}_{\overset{k}{\oplus} E} : \mathcal{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$ given by

$$\begin{array}{rcl} \widetilde{\Delta}_A : \overset{k}{\oplus} E & \to & \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \\ \mathbf{b}_q & \mapsto & \widetilde{\Delta}_A(\mathbf{b}_q) = \xi^{V_A}(pr_A(\mathbf{b}_q), \mathbf{b}_q) = \xi^{V_A}(b_{Aq}, \mathbf{b}_q) \end{array} \quad , \qquad 1 \leq A \leq k \, ,$$

where $\mathbf{b}_q = (b_{1q}, \dots, b_{kq}) \in \overset{k}{\oplus} E$ y $pr_A : \overset{k}{\oplus} E \to E$ is the canonical projection over the A^{th}-copy of E in $\overset{k}{\oplus} E$.

From the local expression (4.12) of ξ^{V_A} and taking into account that

$$y^\alpha(b_{Aq}) = y^\alpha_A(b_{1q}, \dots, b_{kq}) = y^\alpha_A(\mathbf{b}_q)$$

we obtain that $\widetilde{\Delta}_A$ has the following local expression

$$\widetilde{\Delta}_A = \sum_{\alpha=1}^m y^\alpha_A \mathcal{V}^A_\alpha \, , \quad 1 \leq A \leq k \, . \tag{4.13}$$

Remark 4.5. In the standard case, we obtain that each section $\widetilde{\Delta}_A$ turns in the following vector field

$$\begin{array}{rrcl} \Delta_A \colon & T^1_k Q & \to & T(T^1_k Q) \\ & \mathbf{v}_q = (v_{1q}, \dots, v_{Aq}) & \mapsto & (v_{Aq})^{V_A}_{\mathbf{v}_q} \end{array}$$

that is with the A^{th}-canonical vector field on $T^1_k Q$. ◇

In the standard Lagrangian k-symplectic formalism, the canonical vector fields $\Delta_1, \dots, \Delta_k$ allow us to define the energy function. In analogous way, as we will see in the sequel, we will also define de energy function using the Liouville sections $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_k$ in the Lie algebroid setting.

Vertical endomorphism on $\mathcal{T}^E(\overset{k}{\oplus} E)$. The second important family of canonical geometric elements on $\mathcal{T}^E(\overset{k}{\oplus} E)$ is the family of vertical endomorphism $\widetilde{J}^1, \dots, \widetilde{J}^k$.

Definition 4.6. *For each $A = 1, \dots, k$ we define the A^{th}-vertical endomorphism on $\mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$ as the mapping*

$$\begin{array}{rcl} \widetilde{J}^A : \ \mathcal{T}^E(\overset{k}{\oplus} E) & \to & \mathcal{T}^E(\overset{k}{\oplus} E) \\ (e_q, v_{\mathbf{b}_q}) & \mapsto & \widetilde{J}^A(e_q, v_{\mathbf{b}_q}) = \xi^{V_A}(e_q, \mathbf{b}_q) \, , \end{array} \tag{4.14}$$

where $e_q \in E$, $\mathbf{b}_q = (b_{1q}, \dots, b_{kq}) \in \overset{k}{\oplus} E$ and $v_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$.

Lemma 4.7. *Let $\{\mathcal{X}_\alpha, \ \mathcal{V}^A_\alpha\}$ be a local basis of* $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E))$ *and let $\{\mathcal{X}^\alpha, \ \mathcal{V}_A^\alpha\}$ be its dual basis.*

Using this local basis, we obtain that the local expression of $\widetilde{J}^A$ is given by the following expression:

$$\widetilde{J}^A = \sum_{\alpha=1}^m \mathcal{V}^A_\alpha \otimes \mathcal{X}^\alpha \, , \quad 1 \leq A \leq k \, . \tag{4.15}$$

(*Proof*) From (4.3) and (4.12) we obtain

$$\begin{array}{rcl} \widetilde{J}^A(\mathcal{X}_\alpha(\mathbf{b}_q)) & = & \xi^{V_A}(e_\alpha(q),\mathbf{b}_q) = y^\beta(e_\alpha(q))\mathcal{V}^A_\beta(\mathbf{b}_q) = \mathcal{V}^A_\alpha(\mathbf{b}_q)\,, \\ \widetilde{J}^A(\mathcal{V}^B_\alpha(\mathbf{b}_q)) & = & \xi^{V_A}(0_q,\mathbf{b}_q) = 0_{\mathbf{b}_q}\,, \end{array}$$

for each $A, B = 1,\dots,k$, $\alpha = 1\dots,m$, where $\mathbf{b}_q \in \stackrel{k}{\oplus} E$ is an arbitrary element of $\stackrel{k}{\oplus} E$. ■

Remark 4.8.

(i) When one writes the definition of $\widetilde{J}^1,\dots,\widetilde{J}^k$ in the particular case $E = TQ$ and $\rho = id_{TQ}$ one obtains the canonical k-tangent structure $J^1,\dots,J^k$ on T^1_kQ.
(ii) In the particular case $k = 1$ we obtain the vertical endomorphism S on $\mathcal{T}^E(TQ)$, that is, on the prolongation of the Lie algebroid E over $\tau_Q : TQ \to Q$. This endormorphsim S was defined by E. Martínez in [31].

◇

4.1.4. *Second order partial differential equations.* In the standard k-symplectic Lagrangian formalism one obtains the solutions of the Euler-Lagrange equations as integral sections of certain second order partial differential equations (SOPDE) on T^1_kQ .

In order to introduce the analogous objet in the k-symplectic approach on Lie algebroids, now we are going to analyze the concept of SOPDE in the standard case. In this case a SOPDE ξ is a section of the maps

$$\begin{array}{rccc} \tau^k_{T^1_kQ}\colon & T^1_k(T^1_kQ) & \to & T^1_kQ \\ & (v_{1\mathbf{w}_q},\dots,v_{k\mathbf{w}_q}) & \mapsto & \mathbf{w}_q \end{array}$$

and

$$\begin{array}{rccc} T^1_k(\tau^k_Q)\colon & T^1_k(T^1_kQ) & \to & T^1_kQ \\ & (v_{1\mathbf{w}_q},\dots,v_{k\mathbf{w}_q}) & \mapsto & (T_{\mathbf{w}_q}(\tau^k_Q)(v_{1\mathbf{w}_q}),\dots,T_{\mathbf{w}_q}(\tau^k_Q)(v_{k\mathbf{w}_q})) \end{array},$$

where $\tau^k_Q\colon T^1_kQ \to Q$ denotes the canonical projection of the tangent bundle of k^1-velocities.

Returning to our case, we know that: (i) $\stackrel{k}{\oplus} E$ and $\mathcal{T}^E(\stackrel{k}{\oplus} E)$ play the role of T^1_kQ and $T(T^1_kQ)$, respectively; (ii) $T^1_k(T^1_kQ)$ is the Whitney sum of k copies of $T(T^1_kQ)$. Then it is natural to think that the Whitney sum of k copies of $\mathcal{T}^E(\stackrel{k}{\oplus} E)$, that is,

$$(\mathcal{T}^E)^1_k(\stackrel{k}{\oplus} E)\colon = \mathcal{T}^E(\stackrel{k}{\oplus} E)\oplus \stackrel{k}{\dots} \oplus\mathcal{T}^E(\stackrel{k}{\oplus} E)\,,$$

will play the role of $T^1_k(T^1_kQ)$.

Now, the natural question is: *what are the maps playing the role of $\tau^k_{T^1_kQ}$ and $T^1_k(\tau^k_Q)$, when one considers Lie algebroids?*

Now we consider the following maps:

$$\begin{array}{rccc} \widetilde{\tau}^k_{\stackrel{k}{\oplus}E}\colon & (\mathcal{T}^E)^1_k(\stackrel{k}{\oplus} E) \equiv \mathcal{T}^E(\stackrel{k}{\oplus} E)\oplus \stackrel{k}{\dots} \oplus\mathcal{T}^E(\stackrel{k}{\oplus} E) & \to & \stackrel{k}{\oplus} E \\ & ((a_{1q},v_{1\mathbf{b}_q}),\dots,(a_{kq},v_{k\mathbf{b}_q})) & \mapsto & \mathbf{b}_q \end{array}$$

and

$$\begin{array}{rccc} \widetilde{\tau}^k_1\colon & (\mathcal{T}^E)^1_k(\stackrel{k}{\oplus} E) \equiv \mathcal{T}^E(\stackrel{k}{\oplus} E)\oplus \stackrel{k}{\dots} \oplus\mathcal{T}^E(\stackrel{k}{\oplus} E) & \to & \stackrel{k}{\oplus} E \\ & ((a_{1q},v_{1\mathbf{b}_q}),\dots,(a_{kq},v_{k\mathbf{b}_q})) & \mapsto & (a_{1q},\dots,a_{kq}) \end{array}.$$

These two maps play the role of $\tau^k_{T^1_kQ}$ and $T^1_k(\tau^k_Q)$, respectively. In fact, when $E = TQ$ there exists a difeomorphism between $T(T^1_kQ)$ and $\mathfrak{T}^{TQ}(T^1_kQ)$ given by, (see remark 4.2),

$$\begin{array}{rcl} T(T^1_kQ) & \equiv & \mathfrak{T}^{TQ}(T^1_kQ) = (TQ) \times_{TQ} T(T^1_kQ) \equiv T(T^1_kQ) \\ v_{\mathbf{w}_q} & \equiv & (T_{\mathbf{w}_q}(\tau^k_Q)(v_{\mathbf{w}_q}), v_{\mathbf{w}_q}) \end{array} .$$

Thus

- The map
$$\widetilde{\tau}^k_{\overset{k}{\oplus}TQ} \colon (\mathfrak{T}^{TQ})^1_k(T^1_kQ) \equiv T^1_k(T^1_kQ) \to T^1_kQ$$
corresponds to $\tau^k_{T^1_kQ} \colon T^1_k(T^1_kQ) \to T^1_kQ$ since
$$\begin{array}{rl} & \widetilde{\tau}^k_{T^1_kQ}((T_{\mathbf{w}_q}(\tau^k_Q)(v_{1\mathbf{w}_q}), v_{1\mathbf{w}_q}), \ldots, (T_{\mathbf{w}_q}(\tau^k_Q)(v_{k\mathbf{w}_q}), v_{k\mathbf{w}_q})) = \mathbf{w}_q \\ = & \tau^k_{T^1_kQ}(v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}) . \end{array}$$
- The map
$$\widetilde{\tau}_1^k \colon (\mathfrak{T}^{TQ})^1_k(T^1_kQ) \equiv T^1_k(T^1_kQ) \to T^1_kQ$$
identifies with $T^1_k(\tau^k_Q) \colon T^1_k(T^1_kQ) \to T^1_kQ$ since
$$\begin{array}{rl} & \widetilde{\tau}_1^k((T_{\mathbf{w}_q}(\tau^k_Q)(v_{1\mathbf{w}_q}), v_{1\mathbf{w}_q}), \ldots, (T_{\mathbf{w}_q}(\tau^k_Q)(v_{k\mathbf{w}_q}), v_{k\mathbf{w}_q})) \\ = & (T_{\mathbf{w}_q}(\tau^k_Q)(v_{1\mathbf{w}_q}), \ldots, T_{\mathbf{w}_q}(\tau^k_Q)(v_{k\mathbf{w}_q})) = T^1_k(\tau^k_Q)(v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}) . \end{array}$$

Remark 4.9. For simplicity we denote by $(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q})$ an element
$$((a_{1q}, v_{1\mathbf{b}_q}), \ldots, (a_{kq}, v_{k\mathbf{b}_q}))$$
of $(\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E) \equiv \mathfrak{T}^E(\overset{k}{\oplus} E) \oplus \overset{k}{\ldots} \oplus \mathfrak{T}^E(\overset{k}{\oplus} E)$ where $\mathbf{a}_q \colon = (a_{1q}, \ldots, a_{kq}) \in \overset{k}{\oplus} E$ and $\mathbf{v}_{\mathbf{b}_q} \colon = (v_{1\mathbf{b}_q}, \ldots, v_{k\mathbf{b}_q}) \in T^1_k(\overset{k}{\oplus} E)$. ◇

Now we are in conditions to introduce the object which plays the role of a SOPDE when we consider an arbitrary Lie algebroid E. This object is also called SOPDE

Definition 4.10. *A* second order partial differential equation *(*SOPDE *for short) on* $\overset{k}{\oplus} E$ *is a map* $\xi \colon \overset{k}{\oplus} E \to (\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E)$ *which is a section of* $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$ *and* $\widetilde{\tau}_1^k$.

Since $(\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E)$ is the Whitney sum $\mathfrak{T}^E(\overset{k}{\oplus} E) \oplus \overset{k}{\ldots} \oplus \mathfrak{T}^E(\overset{k}{\oplus} E)$ of k copies of $\mathfrak{T}^E(\overset{k}{\oplus} E)$, we deduce that to give a section ξ of $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$ is equivalent to give a family of k sections, $\xi_1, \ldots, \xi_k$, of the Lagrangian prolongation $\mathfrak{T}^E(\overset{k}{\oplus} E)$, obtained by projecting ξ on each factor.

Next, we are going to characterize a SOPDE.

Definition 4.11. *The set*
$$\begin{array}{rcl} Adm(E) & = & \{(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}) \in (\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E) \mid \widetilde{\tau}_1^k(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}) = \widetilde{\tau}^k_{\overset{k}{\oplus}E}(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q})\} \\ & = & \{(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}) \in (\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E) \mid \mathbf{a}_q = \mathbf{b}_q\} . \end{array} \tag{4.16}$$
is called the set of admissible points.

Proposition 4.12. *Let* $\xi = (\xi_1, \ldots, \xi_k) \colon \overset{k}{\oplus} E \to (\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E)$ *be a section of* $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$. *The following statements are equivalent:*

(i) ξ *takes values in* $Adm(E)$.
(ii) ξ *is a* SOPDE, *that is,* $\widetilde{\tau}_1^k \circ \xi = id_{\overset{k}{\oplus} E}$.
(iii) $\widetilde{J}^A(\xi_A) = \widetilde{\Delta}_A$ *for all* $A = 1, \dots, k$.

(*Proof*) From (4.16) it is easy to prove that (i) and (ii) are equivalent. The equivalence between (i) and (iii) is a direct consequence of the definitions of $\widetilde{J}^A$, $\widetilde{\Delta}_A$ and ξ^{V_A}. ∎

Using (*iii*) in Proposition 4.12 one easily deduce that the local expression of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ is the following

$$\xi_A = y_A^\alpha \mathcal{X}_\alpha + (\xi_A)_B^\alpha \mathcal{V}_\alpha^B$$

where $(\xi_A)_B^\alpha \in \mathcal{C}^\infty(\overset{k}{\oplus} E)$.

Proposition 4.13. *Let* $\xi = (\xi_1, \dots, \xi_k) : \overset{k}{\oplus} E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E)$ *be a section of* $\widetilde{\tau}^k_{\overset{k}{\oplus} E}$. *Then*

$$(\rho^{\widetilde{\tau}}(\xi_1), \dots, \rho^{\widetilde{\tau}}(\xi_k)) \colon \; \overset{k}{\oplus} E \to T^1_k(\overset{k}{\oplus} E)$$

is a k-vector field on $\overset{k}{\oplus} E$. *Let us remember that*

$$\rho^{\widetilde{\tau}} : \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \to T(\overset{k}{\oplus} E)$$

denote the anchor map of the Lie algebroid $\mathcal{T}^E(\overset{k}{\oplus} E)$.

(*Proof*) It is a direct consequence of (*vi*) in Section 4.1.2. ∎

In local coordinate we obtain

$$\rho^{\widetilde{\tau}}(\xi_A) = \rho^i_\alpha y^\alpha_A \frac{\partial}{\partial q^i} + (\xi_A)^\alpha_B \frac{\partial}{\partial y^\alpha_B} \in \mathfrak{X}(\overset{k}{\oplus} E) \; . \tag{4.17}$$

Definition 4.14. *A map*

$$\eta : \mathbb{R}^k \to \overset{k}{\oplus} E$$

es called an integral section *of the* SOPDE ξ, *if* η *is an integral section of the k-vector field* $(\rho^{\widetilde{\tau}}(\xi_1), \dots, \rho^{\widetilde{\tau}}(\xi_k))$, *associated to* ξ, *that is,*

$$(\rho^{\widetilde{\tau}}(\xi_A))(\eta(\mathbf{t})) = \eta_*(\mathbf{t}) \left(\frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \right) \; , \quad 1 \le A \le k \; , \tag{4.18}$$

If η is written locally as $\eta(\mathbf{t}) = (\eta^i(\mathbf{t}), \eta^\alpha_A(\mathbf{t}))$, then from (4.17) we deduce that (4.18) is locally equivalent to the identities,

$$\frac{\partial \eta^i}{\partial t^A}\Big|_{\mathbf{t}} = \eta^\alpha_A(\mathbf{t}) \rho^i_\alpha(\widetilde{\tau}(\eta(\mathbf{t})) \; , \quad \frac{\partial \eta^\beta_B}{\partial t^A}\Big|_{\mathbf{t}} = (\xi_A)^\beta_B(\eta(\mathbf{t})) \; , \tag{4.19}$$

where $\widetilde{\tau} : \overset{k}{\oplus} E \to Q$ is the canonical projection.

4.1.5. *Lagrangian formalism.* Let $L : \overset{k}{\oplus} E \to \mathbb{R}$ be a function which we will call Lagrangian function.

In this section, we will develop a intrisic and global geometric framework, which allows us to write the Euler-Lagrange equations on a Lie algebroid, associated with the Lagrangian function L. In first place we are going to introduce some geometric elements associated with a Lagrangian L.

Poincaré-Cartan sections. We now introduce *the Poincaré-Cartan* 1*-sections*

$$\begin{array}{rcl}\Theta_L^A : \stackrel{k}{\oplus} E & \longrightarrow & (\mathcal{T}^E(\stackrel{k}{\oplus} E))^* \\ \mathbf{b}_q & \longmapsto & \Theta_L^A(\mathbf{b}_q)\end{array}$$

where $\Theta_L^A(\mathbf{b}_q)$ is defined by

$$\begin{array}{rrcl}\Theta_L^A(\mathbf{b}_q): & (\mathcal{T}^E(\stackrel{k}{\oplus} E))_{\mathbf{b}_q} & \longrightarrow & \mathbb{R} \\ & Z_{\mathbf{b}_q} = (e_q, v_{\mathbf{b}_q}) & \longmapsto & (\Theta_L^A)_{\mathbf{b}_q}(Z_{\mathbf{b}_q}) = (d^{\mathcal{T}^E(\stackrel{k}{\oplus} E)}L)_{\mathbf{b}_q}((\widetilde{J}^A)_{\mathbf{b}_q}(Z_{\mathbf{b}_q}))\end{array} .$$

Using the expression (4.6) of $d^{\mathcal{T}^E(\stackrel{k}{\oplus} E)}f$ with $f = L$ we obtain:

$$(\Theta_L^A)(\mathbf{b}_q)Z_{\mathbf{b}_q} = (d^{\mathcal{T}^E(\stackrel{k}{\oplus} E)}L)_{\mathbf{b}_q}\Big((\widetilde{J}^A)_{\mathbf{b}_q}Z_{\mathbf{b}_q}\Big) = \Big(\rho^{\widetilde{\tau}}((\widetilde{J}^A)_{\mathbf{b}_q}Z_{\mathbf{b}_q})\Big)L \,, \tag{4.20}$$

where $\mathbf{b}_q \in \stackrel{k}{\oplus} E,\ Z_{\mathbf{b}_q} \in [\mathcal{T}^E(\stackrel{k}{\oplus} E)]_{\mathbf{b}_q}$ y $\rho^{\widetilde{\tau}}((\widetilde{J}^A)_{\mathbf{b}_q}Z_{\mathbf{b}_q}) \in T_{\mathbf{b}_q}(\stackrel{k}{\oplus} E)$.

The Poincaré-Cartan 2-sections

$$\Omega_L^A : \stackrel{k}{\oplus} E \to (\mathcal{T}^E(\stackrel{k}{\oplus} E))^* \wedge (\mathcal{T}^E(\stackrel{k}{\oplus} E))^*,\ 1 \leq A \leq k$$

are defined as follows:

$$\Omega_L^A : = -\mathrm{d}^{\mathcal{T}^E(\stackrel{k}{\oplus} E)}\Theta_L^A \,, \quad 1 \leq A \leq k \,,$$

that is,

$$\begin{array}{rcl}\Omega_L^A(\xi_1,\xi_2) & = & -\mathrm{d}\Theta_L^A(\xi_1,\xi_2) \\ & = & [\rho^{\widetilde{\tau}}(\xi_2)](\Theta_L^A(\xi_1)) - [\rho^{\widetilde{\tau}}(\xi_1)](\Theta_L^A(\xi_2)) + \Theta_L^A([\![\xi_1,\xi_2]\!]^{\widetilde{\tau}}) \,,\end{array} \tag{4.21}$$

where $\xi_1, \xi_2 \in Sec(\mathcal{T}^E(\stackrel{k}{\oplus} E))$ and $(\rho^{\widetilde{\tau}}, [\![\cdot,\cdot]\!]^{\widetilde{\tau}})$ denotes the Lie algebroid structure of $\mathcal{T}^E(\stackrel{k}{\oplus} E)$ defined in section 4.1.2.

Next, we will establish the local expressions of Θ_L^A and Ω_L^A.

Consider $\{\mathcal{X}_\alpha,\ \mathcal{V}_\alpha^B\}$ a local basis of sections of $\mathrm{Sec}(\mathcal{T}^E(\stackrel{k}{\oplus} E))$ and $\{\mathcal{X}^\alpha,\ \mathcal{V}^\alpha_B\}$ its dual basis. Then from (4.4), (4.15) y (4.20) we obtain

$$\Theta_L^A = \frac{\partial L}{\partial y_A^\alpha}\mathcal{X}^\alpha \,, \qquad 1 \leq A \leq k \,. \tag{4.22}$$

From de local expressions (4.3), (4.4), (4.5), (4.21) and (4.22) we have for each $A = 1, \dots, k$,

$$\Omega_L^A = \frac{1}{2}\left(\rho_\beta^i \frac{\partial^2 L}{\partial q^i \partial y_A^\alpha} - \rho_\alpha^i \frac{\partial^2 L}{\partial q^i \partial y_A^\beta} + \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial L}{\partial y_A^\gamma}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{\partial^2 L}{\partial y_B^\beta \partial y_A^\alpha}\,\mathcal{X}^\alpha \wedge \mathcal{V}_B^\beta \,. \tag{4.23}$$

Remark 4.15.

(i) En the particular case $k = 1$ we obtain the Poincaré-Cartan forms of the Lagrangian Mechanics on Lie algebroids. See, for instance [8, 31].

(ii) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$, then

$$\Omega_L^A(X,Y) = \omega_L^A(X,Y) \,, \quad 1 \leq A \leq k$$

where X, Y are two vector fields on $T_k^1 Q$ and $\omega_L^1, \dots, \omega_L^k$ denote the Lagrangian 2-forms of the standard k-symplectic formalism defined by $\omega_L^A = -d(dL \circ J^A)$, being d the usual differential.

$\diamond$

The energy function. The *energy function* $E_L : \overset{k}{\oplus} E \to \mathbb{R}$ defined by the Lagrangian L is

$$E_L = \sum_{A=1}^{k} \rho^{\widetilde{\tau}}(\Delta_A)L - L \,,$$

and from (4.4) and (4.13) one deduce that E_L is locally given by

$$E_L = \sum_{A=1}^{k} y_A^\alpha \frac{\partial L}{\partial y_A^\alpha} - L \,. \tag{4.24}$$

Morphisms. For studying the concept of Euler-Lagrange equations and their solutions on Lie algebroids, we need to show a new point of view of the solutions for the standard Euler-Lagrange equations, which allows us to think a solution as a particular set of Lie algebroid morphisms.

In the standard Lagrangian k-symplectic formalism, a solution of the Euler-Lagrange equation is a field $\phi : \mathbb{R}^k \to Q$ such that its first prolongation $\phi^{(1)} : \mathbb{R}^k \to T^1_k Q$ satisfies the Euler-Lagrange field equations, that is,

$$\sum_{A=1}^{k} \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} .$$

Let us observe that the map ϕ naturally induces the following Lie algebroid morphism

$$\begin{tikzcd} T\mathbb{R}^k \arrow[r, "T\phi"] \arrow[d, "\tau_{\mathbb{R}^k}"'] & TQ \arrow[d, "\tau_Q"] \\ \mathbb{R}^k \arrow[r, "\phi"'] & Q \end{tikzcd}$$

If we consider the canonical basis of section of $\tau_{\mathbb{R}^k}$, $\left\{ \dfrac{\partial}{\partial t^1}, \dots, \dfrac{\partial}{\partial t^k} \right\}$, then the first prolongation $\phi^{(1)}$ of ϕ, can be written as follows:

$$\phi^{(1)}(\mathbf{t}) = (T_{\mathbf{t}}\phi(\frac{\partial}{\partial t^1}\Big|_{\mathbf{t}}), \dots, T_{\mathbf{t}}\phi(\frac{\partial}{\partial t^k}\Big|_{\mathbf{t}})) \,.$$

Returning to the case of Lie algebroids, the analog of a field solution of the Euler-Lagrange equations is now a Lie algebroid morphism $\Phi = (\overline{\Phi}, \underline{\Phi})$ between $T\mathbb{R}^k$ and E

$$\begin{tikzcd} T\mathbb{R}^k \arrow[r, "\overline{\Phi}"] \arrow[d, "\tau_{\mathbb{R}^k}"'] & E \arrow[d, "\tau"] \\ \mathbb{R}^k \arrow[r, "\underline{\Phi}"'] & Q \end{tikzcd}$$

Taking a local basis $\{e_A\}_{A=1}^k$ of local sections of $T\mathbb{R}^k$, one can define a map $\widetilde{\Phi} : \mathbb{R}^k \to \overset{k}{\oplus} E$ associated to Φ and given by

$$\begin{array}{rcl} \widetilde{\Phi} : \mathbb{R}^k & \to & \overset{k}{\oplus} E \equiv E \oplus \overset{k}{\dots} \oplus E \\ \mathbf{t} & \to & (\overline{\Phi}(e_1(\mathbf{t})), \dots, \overline{\Phi}(e_k(\mathbf{t}))) \,. \end{array}$$

Let (t^A) and (q^i) be a local coordinate system on $\mathbb{R}^k$ and Q, respectively. Let $\{e_A\}$ be a local basis of sections of $\tau_{\mathbb{R}^k}$ and $\{e_\alpha\}$ be a local basis of sections of E, we denote by $\{e^A\}$ and $\{e^\alpha\}$ the dual basis. Then Φ is determined by the relations $\underline{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}))$ and

$\Phi^* e^\alpha = \phi^\alpha_A e^A$ for certain local functions ϕ^i and ϕ^α_A on $\mathbb{R}^k$. Thus, the associated map $\widetilde{\Phi}$ is locally given by $\widetilde{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}), \phi^\alpha_A(\mathbf{t}))$.

In this case, the conditions of Lie algebroid morphism (3.6) are written as

$$\rho^i_\alpha \phi^\alpha_A = \frac{\partial \phi^i}{\partial t^A} \quad , \quad 0 = \frac{\partial \phi^\alpha_A}{\partial t^B} - \frac{\partial \phi^\alpha_B}{\partial t^A} + \mathcal{C}^\alpha_{\beta\gamma}\phi^\beta_B \phi^\alpha_A \,. \tag{4.25}$$

Remark 4.16. In the standard case where $E = TQ$ the above morphism conditions reduce to

$$\phi^i_A = \frac{\partial \phi^i}{\partial t^A} \quad \text{and} \quad \frac{\partial \phi^i_A}{\partial t^B} = \frac{\partial \phi^i_B}{\partial t^A}\,.$$

Therefore, in the standard case, by considering morphisms we are just considering the first-order prolongation of the fields $\phi : \mathbb{R}^k \to Q$. $\diamond$

The Euler-Lagrange equations. Consider a given regular Lagrangian function $L\colon \overset{k}{\oplus} E \to \mathbb{R}$. The field equations are obtained as follows:

We look for the solutions $\xi = (\xi_1, \dots, \xi_k)$ of the equation

$$\sum_{A=1}^{k} \imath_{\xi_A} \Omega^A_L = \mathrm{d}^{\mathfrak{T}^E(\overset{k}{\oplus} E)} E_L \,. \tag{4.26}$$

Notice that each ξ_A is a section of the Lagrangian prolongation $\mathfrak{T}^E(\overset{k}{\oplus} E)$ and thus, ξ is a section of $(\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E) = \mathfrak{T}^E(\overset{k}{\oplus} E) \oplus \overset{k}{\ldots} \oplus \mathfrak{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$.

Using a local coordinate system (q^i, y^α_A) on $\overset{k}{\oplus} E$ an a local basis $\{e_\alpha\}$ of $\mathrm{Sec}(E)$, each ξ_A is locally given by

$$\xi_A = \xi^\alpha_A \mathcal{X}_\alpha + (\xi_A)^\alpha_C \mathcal{V}^C_\alpha \,.$$

Then, using this local expression and from (4.6), (4.23) and (4.24) we obtain that the equation (4.26) is locally expressed as follows:

$$\begin{aligned} \xi^\beta_A\Big(\rho^i_\alpha \frac{\partial^2 L}{\partial q^i \partial y^\beta_A} - \rho^i_\beta \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A} + \mathcal{C}^\gamma_{\beta\alpha}\frac{\partial L}{\partial y^\gamma_A}\Big) - (\xi_A)^\beta_B \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A} &= \rho^i_\alpha\Big(y^\beta_A \frac{\partial^2 L}{\partial q^i \partial y^\beta_A} - \frac{\partial L}{\partial q^i}\Big), \\ \xi^\alpha_A \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A} &= y^\alpha_A \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A}\,. \end{aligned}$$

Since L is regular, that is the matrix $(\frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B})$ is regular, the above equations can be written as follows

$$\begin{aligned} y^\beta_A \rho^i_\beta \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A} + (\xi_A)^\beta_B \frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B} &= \rho^i_\alpha \frac{\partial L}{\partial q^i} + y^\beta_A \mathcal{C}^\gamma_{\beta\alpha}\frac{\partial L}{\partial y^\gamma_A}\,, \\ \xi^\alpha_A &= y^\alpha_A\,. \end{aligned} \tag{4.27}$$

Therefore ξ is a SOPDE.

Let $\widetilde{\Phi}\colon \mathbb{R}^k \to \overset{k}{\oplus} E$ the associated map to a Lie algebroid morphism $\Phi\colon T\mathbb{R}^k \to E$.

If $\widetilde{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}), \phi^\alpha_A(\mathbf{t}))$ is an integral section of the SOPDE ξ solution of (4.26) then from the condition (4.19) of integral section and the equations (4.27) we obtain

$$\begin{aligned}
\frac{\partial \phi^i}{\partial t^A}\Big|_{\mathbf{t}} \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A}\Big|_{\widetilde{\Phi}(\mathbf{t})} + \frac{\partial \phi^\beta_B}{\partial t^A}\Big|_{\mathbf{t}} \frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B}\Big|_{\widetilde{\Phi}(\mathbf{t})} &= \rho^i_\alpha \frac{\partial L}{\partial q^i}\Big|_{\widetilde{\Phi}(\mathbf{t})} + \phi^\beta_A \mathcal{C}^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_A}\Big|_{\widetilde{\Phi}(\mathbf{t})} , \\
\frac{\partial \phi^i}{\partial t^A}\Big|_{\mathbf{t}} &= \rho^i_\alpha \phi^\alpha_A(\mathbf{t}) , \\
0 &= \frac{\partial \phi^\alpha_A}{\partial t^B}\Big|_{\mathbf{t}} - \frac{\partial \phi^\alpha_B}{\partial t^A}\Big|_{\mathbf{t}} + \mathcal{C}^\alpha_{\beta\gamma} \phi^\beta_B(\mathbf{t}) \phi^\gamma_A(\mathbf{t})
\end{aligned}$$

where the later equation is a consequence of the morphism condition (4.25). The above equations can be written as follows:

$$\begin{aligned}
\sum_{A=1}^k \frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial y^\alpha_A}\Big|_{\widetilde{\Phi}(\mathbf{t})} \right) &= \rho^i_\alpha \frac{\partial L}{\partial q^i}\Big|_{\widetilde{\Phi}(\mathbf{t})} + \phi^\beta_C \mathcal{C}^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_C}\Big|_{\widetilde{\Phi}(\mathbf{t})} \\
\frac{\partial \phi^i}{\partial t^A}\Big|_{\mathbf{t}} &= \rho^i_\alpha \phi^\alpha_A(\mathbf{t}) , \\
0 &= \frac{\partial \phi^\alpha_A}{\partial t^B}\Big|_{\mathbf{t}} - \frac{\partial \phi^\alpha_B}{\partial t^A}\Big|_{\mathbf{t}} + \mathcal{C}^\alpha_{\beta\gamma} \phi^\beta_B(\mathbf{t}) \phi^\gamma_A(\mathbf{t}) .
\end{aligned} \tag{4.28}$$

Notice that if E is the standard Lie algebroid TQ then the above equations are the classical Euler-Lagrange equations in field theories for the Lagrangian $L : T^1_k Q \to \mathbb{R}$. Thus, in the sequel, (4.28) will be called the *Euler-Lagrange equations for field theories on Lie algebroids.*

Remark 4.17.

(i) The equations (4.28) are obtained by E. Martinez from a variational approach in the multisymplectic framework, see [33].
(ii) If one rewrite the above equations in the particular case, $k = 1$, one obtain the Euler-Lagrange equations on Lie algebroids given by Weinstein in [47].
(iii) When $E = TQ$, the equations (4.28) coincides with the Euler- Lagrange equations of the Günther formalism, [36].

◇

The results of this section can be summarized in the following

Theorem 4.18. *Let $L : \mathbb{R}^k \to \stackrel{k}{\oplus} E$ be a regular Lagrangian and $\xi_1, \dots, \xi_k$ be k sections of $\widetilde{\tau}_{\stackrel{k}{\oplus} E} : \mathcal{T}^E(\stackrel{k}{\oplus} E) \to \stackrel{k}{\oplus} E$ such that*

$$\sum_{A=1}^k \imath_{\xi_A} \Omega^A_L = \mathrm{d}^{\mathcal{T}^E(\stackrel{k}{\oplus} E)} E_L .$$

Then:

(i) $\xi = (\xi_1, \dots, \xi_k)$ *is a* SOPDE.
(ii) *Let $\widetilde{\Phi} : \mathbb{R}^k \to \stackrel{k}{\oplus} E$ be the map associated with a Lie algebroid morphism between $T\mathbb{R}^k$ and E. If $\widetilde{\Phi}$ is an integral section of ξ, then it is a solution of the Euler-Lagrange equations for field theories on Lie algebroids (4.28).*

Remark 4.19. If we rewrite this section in the particular case $k = 1$, we reobtain the Lagrangian Mechanics on a Lie algebroid. (See section 3.1 in [8] or section 2.2 in [19]). ◇

As a final remark in this subsection, it is interesting to point out that the standard Lagrangian k-symplectic formalism is a particular case of the Lagrangian formalism on Lie algebroids, when $E = TQ$, the anchor map ρ_{TQ} is the identity on TQ and the structure constants are $\mathcal{C}^\gamma_{\alpha\beta} = 0$.

In this case we have:

- The manifold $\stackrel{k}{\oplus} E$ identifies with T^1_kQ, $\mathcal{T}^{TQ}(T^1_kQ)$ with $T(T^1_kQ)$ and $(\mathcal{T}^{TQ})^1_k(T^1_kQ)$ with $T^1_k(T^1_kQ)$.
- The energy function $E_L : T^1_kQ \to \mathbb{R}$ is given by $E_L = \displaystyle\sum_{A=1}^k \Delta_A(L) - L$ where Δ_A are the canonical vector field on T^1_kQ. We have explained how to obtain this vector fields in Remark 4.5 .
- A section $\xi :\stackrel{k}{\oplus} E \to (\mathcal{T}^E)^1_k(\stackrel{k}{\oplus} E)$ correspond to a k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on T^1_kQ, that is, ξ is a section of $\tau^k_{T^1_kQ} : T^1_k(T^1_kQ) \to T^1_kQ$.
- A SOPDE ξ is a k-vector field on T^1_kQ which is a section of $T^1_k(\tau^k_Q) : T^1_k(T^1_kQ) \to T^1_kQ$.
- Let f be a function on T^1_kQ then
$$\mathrm{d}^{\mathcal{T}^{TQ}(T^1_kQ)} f(Y) = df(Y)\,,$$
where df denotes the standard differential and Y is a vector field on T^1_kQ.
- It is satisfies that
$$\Omega^A_L(X, Y) = \omega^A_L(X, Y), \qquad A = 1, \dots, k$$
where ω^A_L, $A = 1, \dots, k$ are the Lagrangian 2-forms of the standard k-symplectic formalism given by $\omega^A_L = -d(dL \circ J^A)$.
- Thus, in the standard k-symplectic Lagrangian formalism, the equation (4.26) can be written as follow:
$$\sum_{A=1}^k \imath_{\xi_A} \omega^A_L = dE_L\,,$$
that is, this equation is the geometric Euler-Lagrange equations in the standard k-symplectic Lagrangian formalism.
- In the standard case a map $\phi : \mathbb{R}^k \to Q$ induces a Lie algebroid morphism $(T\phi, \phi)$ between $T\mathbb{R}^k$ and TQ. In this case, the associated map $\widetilde{\Phi}$ of this morphism is the first prolongation $\phi^{(1)}$ of ϕ given by
$$\widetilde{\Phi}(\mathbf{t}) = (T\phi(\frac{\partial}{\partial t^1}\Big|_{\mathbf{t}}), \dots, T\phi(\frac{\partial}{\partial t^k}\Big|_{\mathbf{t}}))\,.$$
Let us observe that $\widetilde{\Phi} = \phi^{(1)}$ (see 2.2).

Thus, from the Theorem 4.18 and the above remarks, we deduce the following corollary which summarizes the standard Lagrangian k-symplectic formalism, see [14, 36, 41].

Corollary 4.20. *Let $L : T^1_kQ \to \mathbb{R}$ be a regular Lagrangian and $\xi = (\xi_1, \dots, \xi_k)$ a k-vector field on T^1_kQ such that*
$$\sum_{A=1}^k i_{\xi_A} \omega^A_L = dE_L\,.$$
Then:

(i) *ξ is a* SOPDE

(ii) *If $\widetilde{\Phi} \equiv \phi^{(1)}$ is an integral section of the k-vector field ξ, then it is a solution of the Euler-Lagrange field equations in the standard Lagrangian k-symplectic field theories given by*

$$\sum_{A=1}^{k} \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\widetilde{\Phi}(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i}\Big|_{\widetilde{\Phi}(\mathbf{t})} \quad , \quad v^i_A(\widetilde{\Phi}(\mathbf{t})) = \frac{\partial (q^i \circ \widetilde{\Phi})}{\partial t^A}\Big|_{\mathbf{t}} .$$

4.2. **Hamiltonian formalism.** In this subsection we will develop the Hamiltonian k-symplectic formalism on Lie algebroids, in an analogous way that in the standard case

Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ be a Lie algebroid over a manifold Q. For the Hamiltonian approach we consider the dual bundle, $\tau^* : E^* \to Q$ of E.

4.2.1. *The manifold $\stackrel{k}{\oplus} E^*$.* The standard k-symplectic Hamiltonian formalism develops on the bundle $(T^1_k)^*Q$ of k^1-covelocities of Q, that is, the Whitney sum of k copies of T^*Q. Passing to Lie algebroids E as a substitute of the tangent bundle, its natural to think that the analog of $(T^1_k)^*Q$ is the Whitney sum over Q of k copies of the dual space E^*.

We denote by

$$\stackrel{k}{\oplus} E^* = E^* \oplus \stackrel{k}{\dots} \oplus E^* ,$$

the Whitney sum of k copies of the vector bundle E^*, the projection map

$$\widetilde{\tau}^* : \stackrel{k}{\oplus} E^* \to Q,$$

given by $\widetilde{\tau}^*(a^*_{1_q}, \dots, a^*_{k_q}) = q$

If (q^i, y_α) are local coordinates on $(\tau^*)^{-1}(U) \subseteq E^*$, then the induced local coordinates (q^i, y^A_α) on $(\widetilde{\tau}^*)^{-1}(U) \subseteq \stackrel{k}{\oplus} E^*$ are given by

$$q^i(a^*_{1_q}, \dots, a^*_{k_q}) = q^i(q) , \quad y^A_\alpha(a^*_{1_q}, \dots, a^*_{k_q}) = y_\alpha(a^*_{A_q}) .$$

4.2.2. *The Hamiltonian prolongation.* For the description of the Hamiltonian k-symplectic formalism on Lie algebroids we consider the prolongation of a Lie algebroid E over the fibration $\widetilde{\tau}^* \colon \stackrel{k}{\oplus} E^* \to Q$, that is, (see Section 3.4),

$$\mathcal{T}^E(\stackrel{k}{\oplus} E^*) = \{(e_q, v_{\mathbf{b}^*_q}) \in E \times T(\stackrel{k}{\oplus} E^*) / \ \rho_E(e_q) = T(\widetilde{\tau}^*)(v_{\mathbf{b}^*_q})\} . \tag{4.29}$$

Taking into account the description of the prolongation $\mathcal{T}^E P$, (see for instance, [8, 19, 30] or section 3.4 in this paper), on the particular case $P = E^* \oplus \stackrel{k}{\dots} \oplus E^*$ we obtain:

(i) $\mathcal{T}^E(\stackrel{k}{\oplus} E^*) \equiv E \times_{TQ} T(\stackrel{k}{\oplus} E^*)$ is a Lie algebroid over $\stackrel{k}{\oplus} E^*$ with projection

$$\widetilde{\tau}_{\stackrel{k}{\oplus} E^*} : \mathcal{T}^E(\stackrel{k}{\oplus} E^*) \equiv E \times_{TQ} T(\stackrel{k}{\oplus} E^*) \longrightarrow \stackrel{k}{\oplus} E^*$$

and Lie algebroid structure $([\![\cdot,\cdot]\!]^{\widetilde{\tau}^*}, \rho^{\widetilde{\tau}^*})$ where the anchor map

$$\rho^{\widetilde{\tau}^*} = E \times_{TQ} T(\stackrel{k}{\oplus} E^*) \colon \mathcal{T}^E(\stackrel{k}{\oplus} E^*) \to T(\stackrel{k}{\oplus} E^*)$$

is the canonical projection over the second factor.

We refer to this Lie algebroid as the *Hamiltonian prolongation.*

(ii) If (q^i, y^A_α) denotes a local coordinate system of $\stackrel{k}{\oplus} E^*$ then the induced local coordinate system on $\mathcal{T}^E(\stackrel{k}{\oplus} E^*) \equiv E \times_{TQ} T(\stackrel{k}{\oplus} E^*)$ is given by

$$(q^i, y^A_\alpha, z^\alpha, w^A_\alpha)_{1 \le i \le n,\, 1 \le A \le k,\, 1 \le \alpha \le m}$$

where

$$
\begin{array}{rclcrcl}
q^i(e_q, v_{\mathbf{b}^*_q}) &=& q^i(q)\,, && y^A_\alpha(e_q, v_{\mathbf{b}^*_q}) &=& y^A_\alpha(\mathbf{b}^*_q)\,,\\
z^\alpha(e_q, v_{\mathbf{b}^*_q}) &=& y^\alpha(e_q)\,, && w^A_\alpha(e_q, v_{\mathbf{b}^*_q}) &=& v_{\mathbf{b}^*_q}(y^A_\alpha)\,.
\end{array}
\tag{4.30}
$$

(iii) The set $\{\mathcal{X}_\alpha, \mathcal{V}^\alpha_A\}$ given by

$$
\begin{array}{rccccl}
\mathcal{X}_\alpha\colon & \stackrel{k}{\oplus} E^* & \to & \mathcal{T}^E(\stackrel{k}{\oplus} E^*) & \equiv & E\times_{TQ} T(\stackrel{k}{\oplus} E^*)\\
& \mathbf{b}^*_q & \mapsto & \mathcal{X}_\alpha(\mathbf{b}^*_q) & = & (e_\alpha(q); \rho^i_\alpha(q)\dfrac{\partial}{\partial q^i}\Big|_{\mathbf{b}^*_q})\\
\\
\mathcal{V}^\alpha_A\colon & \stackrel{k}{\oplus} E^* & \to & \mathcal{T}^E(\stackrel{k}{\oplus} E^*) & \equiv & E\times_{TQ} T(\stackrel{k}{\oplus} E^*)\\
& \mathbf{b}^*_q & \mapsto & \mathcal{V}^\alpha_A(\mathbf{b}^*_q) & = & (0_q; \dfrac{\partial}{\partial y^A_\alpha}\Big|_{\mathbf{b}^*_q})\,,
\end{array}
\tag{4.31}
$$

is a local basis of $Sec(\mathcal{T}^E(\stackrel{k}{\oplus} E^*))$ the set of sections of $\widetilde{\tau}_{\stackrel{k}{\oplus} E^*}$. (See (3.7)).

(iv) The anchor map $\rho^{\widetilde{\tau}^*}\colon \mathcal{T}^E(\stackrel{k}{\oplus} E^*)\to T(\stackrel{k}{\oplus} E^*)$ allows us to associate a vector field to each section $\xi\colon \stackrel{k}{\oplus} E^*\to \mathcal{T}^E(\stackrel{k}{\oplus} E^*)$ of $\widetilde{\tau}_{\stackrel{k}{\oplus} E^*}$.

Locally, if ξ writes as follows:

$$\xi = \xi^\alpha \mathcal{X}_\alpha + \xi^A_\alpha \mathcal{V}^\alpha_A \in Sec(\mathcal{T}^E(\stackrel{k}{\oplus} E^*))$$

then the associated vector field has the following local expression, see (3.8),

$$\rho^{\widetilde{\tau}^*}(\xi) = \rho^i_\alpha \xi^\alpha \frac{\partial}{\partial q^i} + \xi^A_\alpha \frac{\partial}{\partial y^A_\alpha} \in \mathfrak{X}(\stackrel{k}{\oplus} E^*)\,. \tag{4.32}$$

(v) The Lie bracket of two sections of $\widetilde{\tau}_{\stackrel{k}{\oplus} E^*}$ is characterized by the following expressions (see (3.10)):

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\widetilde{\tau}^*} = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \quad [\![\mathcal{X}_\alpha, \mathcal{V}^\beta_B]\!]^{\widetilde{\tau}^*} = 0 \quad [\![\mathcal{V}^\alpha_A, \mathcal{V}^\beta_B]\!]^{\widetilde{\tau}^*} = 0\,, \tag{4.33}$$

(vi) If $\{\mathcal{X}^\alpha, \mathcal{V}^A_\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}^\alpha_A\}$, then the exterior differential is determined by, (see (3.11)),

$$
\begin{array}{rcl}
d^{\mathcal{T}^E(\stackrel{k}{\oplus} E^*)} f &=& \rho^i_\alpha \dfrac{\partial f}{\partial q^i}\mathcal{X}^\alpha + \dfrac{\partial f}{\partial y^A_\alpha}\mathcal{V}^A_\alpha\,, \quad \text{for all } f\in \mathcal{C}^\infty(\stackrel{k}{\oplus} E^*)\\
d^{\mathcal{T}^E(\stackrel{k}{\oplus} E^*)}\mathcal{X}^\gamma &=& -\dfrac{1}{2}\mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha\wedge\mathcal{X}^\beta, \quad d^{\mathcal{T}^E(\stackrel{k}{\oplus} E^*)}\mathcal{V}^A_\gamma = 0\,.
\end{array}
\tag{4.34}
$$

Remark 4.21. In the particular case $E = TQ$ the manifold $\mathcal{T}^E(\stackrel{k}{\oplus} E^*)$ turns into $T((T^1_k)^*Q)$.The proof is similar to remark 4.2. $\diamond$

4.2.3. *The vector bundle* $\mathcal{T}^E(\stackrel{k}{\oplus} E^*)\oplus \stackrel{k}{\dots} \oplus\mathcal{T}^E(\stackrel{k}{\oplus} E^*)$. In the standard k-symplectic Hamiltonian formalism one obtains the solutions of the Hamilton equations as integral sections of certain k-vector fields on $(T^1_k)^*Q$, that is, certain sections of

$$\tau^k_{(T^1_k)^*Q}\colon T^1_k((T^1_k)^*Q)\to (T^1_k)^*Q\,.$$

Thinking on a Lie algebroid E as a substitute of the tangent bundle, we know that $\mathcal{T}^E(\stackrel{k}{\oplus} E^*)$ plays the role of $T((T^1_k)^*Q)$. Thus it is natural to choose the Whitney sum of

k copies of $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, that is, the manifold

$$(\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) \colon = \mathcal{T}^E(\overset{k}{\oplus} E^*)\oplus \overset{k}{\ldots} \oplus\mathcal{T}^E(\overset{k}{\oplus} E^*)$$

plays the role of

$$T^1_k((T^1_k)^*Q) = T((T^1_k)^*Q)\oplus \overset{k}{\ldots} \oplus T((T^1_k)^*Q)\,.$$

We denote by $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*} : (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) \to \overset{k}{\oplus} E^*$ the canonical projection given by

$$\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}(Z^1_{\mathbf{b}^*_q}, \dots, Z^k_{\mathbf{b}^*_q}) = \mathbf{b}^*_q\,,$$

where $Z^A_{\mathbf{b}^*_q} = (a_{Aq}, v_{A\mathbf{b}^*_q}) \in \mathcal{T}^E(\overset{k}{\oplus} E^*),\ A = 1, \dots, k.$

Now, we consider a section ξ of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$. Next we will prove that there exist a k-vector field on $\overset{k}{\oplus} E^*$ associated to each section ξ.

Notice that to give a section

$$\xi\colon\ \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*)) = \mathcal{T}^E(\overset{k}{\oplus} E^*)\oplus \overset{k}{\ldots} \oplus\mathcal{T}^E(\overset{k}{\oplus} E^*)$$

of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$ is equivalent to give k sections, $\xi_1, \dots, \xi_k$ of the Hamiltonian prolongation $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, obtained by projection ξ on each factor $\mathcal{T}^E(\overset{k}{\oplus} E^*)$.

Proposition 4.22. *Let $\xi = (\xi^1, \dots, \xi^k)$ be a section of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$. Then*

$$(\rho^{\widetilde{\tau}^*}(\xi_1), \dots, \rho^{\widetilde{\tau}^*}(\xi_k))\colon\ \overset{k}{\oplus} E^* \to T^1_k(\overset{k}{\oplus} E^*)$$

is a k-vector field on $\overset{k}{\oplus} E^*$. Let us remember that the mapping $\rho^{\widetilde{\tau}^*}$ is the anchor map of the Lie algebroid $\mathcal{T}^E(\overset{k}{\oplus} E^*)$.

(*Proof*) Is a direct consequence of (4.32) and the above remark. ∎

4.2.4. *Hamiltonian formalism.* Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ be a Lie algebroid on a manifold Q and $H :\overset{k}{\oplus} E^* \to \mathbb{R}$ be a Hamiltonian function.

In this subsection we will develop the k-symplectic Hamiltonian formalism on Lie algebroids. Morevover, we also describe the standard k-symplectic Hamiltonian formalism as a particular case of the formalism developed here.

First, we define certain type of sections of the dual of the Hamiltonian prolongation $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, which play the role of the Liouville forms in the standard case.

The Liouville sections. We are called *Liouville* 1*-sections* to the sections of the bundle $(\mathcal{T}^E(\overset{k}{\oplus} E^*))^* \to\overset{k}{\oplus} E^*$ defined as follow:

$$\begin{array}{rcl} \Theta^A :\overset{k}{\oplus} E^* & \longrightarrow & (\mathcal{T}^E(\overset{k}{\oplus} E^*))^* \\ \mathbf{b}^*_q & \longmapsto & \Theta^A_{\mathbf{b}_q^*} \end{array} \qquad 1 \leq A \leq k\,,$$

where $\Theta^A_{\mathbf{b}_q^*}$ is the function given by

$$\begin{array}{rrcl} \Theta^A_{\mathbf{b}_q^*} : & (\mathcal{T}^E(\overset{k}{\oplus} E^*))_{\mathbf{b}_q^*} & \longrightarrow & \mathbb{R} \\ & (e_q, v_{\mathbf{b}_q^*}) & \longmapsto & \Theta^A_{\mathbf{b}_q^*}(e_q, v_{\mathbf{b}_q^*}) = b^*_{A_q}(e_q)\ , \end{array} \tag{4.35}$$

for each $e_q \in E$, $\mathbf{b}^*_q = ({b_1}^*_q, \dots, {b_k}^*_q) \in \overset{k}{\oplus} E^*$ and $v_{\mathbf{b}^*_q} \in T_{\mathbf{b}^*_q}(\overset{k}{\oplus} E^*)$.

Now we define the 2-sections

$$\Omega^A : \overset{k}{\oplus} E^* \to (\mathcal{T}^E(\overset{k}{\oplus} E^*))^* \wedge (\mathcal{T}^E(\overset{k}{\oplus} E^*))^*, \ 1 \leq A \leq k$$

by

$$\Omega^A = -d^{\mathcal{T}^E(\overset{k}{\oplus} E^*)}\Theta^A ,$$

where $d^{\mathcal{T}^E(\overset{k}{\oplus} E^*)}$ denotes the exterior differential on the Lie algebroid $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, see (4.34).

Next we will write the local expression of the sections Θ^A and Ω^A.

Let $\{\mathcal{X}_\alpha, \ \mathcal{V}^\beta_B\}$ be a local basis of $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E^*))$ and $\{\mathcal{X}^\alpha_A, \ \mathcal{V}^B_\beta\}$ its dual basis. Then from (4.31) we have

$$\Theta^A = \sum_{\beta=1}^{m} y^A_\beta \mathcal{X}^\beta , \quad 1 \leq A \leq k . \tag{4.36}$$

Thus, from (4.32), (4.33), (4.34) and (4.36) we obtain the local expression of Ω^A, that is,

$$\Omega^A = \sum_\beta \mathcal{X}^\beta \wedge \mathcal{V}^A_\beta + \frac{1}{2}\sum_{\beta,\gamma,\delta} \mathcal{C}^\delta_{\beta\gamma} y^A_\delta \mathcal{X}^\beta \wedge \mathcal{X}^\gamma , \quad 1 \leq A \leq k . \tag{4.37}$$

Remark 4.23.

(i) In the particular case $k = 1$ the Liouville sections introduced here are the Liouville sections on Mechanics on Lie algebroids, see E. Martinez, [8, 31].

(ii) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$, then

$$\Omega^A(X, Y) = \omega^A(X, Y) , \qquad 1 \leq A \leq k$$

where X, Y are vector fiels on $(T^1_k)^*Q$ and $\omega^1, \dots, \omega^k$ are the canonical 2-forms of the standard k-symplectic Hamiltonian formalism.

$\diamond$

The Hamilton equations.

Theorem 4.24. *Let $H : \overset{k}{\oplus} E^* \to \mathbb{R}$ be a Hamiltonian and*

$$\xi = (\xi_1, \dots, \xi_k) : \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) \equiv \mathcal{T}^E(\overset{k}{\oplus} E^*) \oplus \overset{k}{\dots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E^*)$$

a section of $\widetilde{\tau}^k_{\overset{k}{\oplus}E^*}$, or equivalently, k sections of the Hamiltonian prolongation, $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, such that

$$\sum_{A=1}^{k} \imath_{\xi_A}\Omega^A = \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} H . \tag{4.38}$$

If $\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^*$ is an integral section of ξ, then ψ is a solution of the following system of partial differential equations

$$\frac{\partial \psi^i}{\partial t^A} = \rho^i_\alpha \frac{\partial H}{\partial y^A_\alpha} \quad \text{and} \quad \sum_{A=1}^{k} \frac{\partial \psi^A_\alpha}{\partial t^A} = -\left(\mathcal{C}^\delta_{\alpha\beta}\, \psi^B_\delta \, \frac{\partial H}{\partial y^B_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right) . \tag{4.39}$$

Remark 4.25. In the particular case $E = TQ$ and $\rho = id_{TQ}$, the equations (4.2.4) are the Hamilton field equations. Therefore these equations (4.2.4) are called *the Hamilton equations on Lie algebroids.* $\diamond$

(*Proof*) Let

$$\xi = (\xi_1, \ldots, \xi_k) : \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*)$$

be a section of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$ such that (4.38) holds.

Consider $\{\mathcal{X}_\alpha,\ \mathcal{V}^\beta_B\}$, a local basis of sections of $\widetilde{\tau}_{\overset{k}{\oplus} E^*} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to \overset{k}{\oplus} E^*$, then each $\xi_A,\ A = 1, \ldots, k$ can be written as follow:

$$\xi_A = \xi^\alpha_A \mathcal{X}_\alpha + (\xi_A)^B_\alpha \mathcal{V}^\alpha_B \ , \tag{4.40}$$

From (4.34), (4.37) and (4.40) we obtain that (4.38) is locally expressed as follow

$$\begin{aligned} \xi^\alpha_B &= \frac{\partial H}{\partial y^B_\alpha} \\ \sum_{A=1}^k (\xi_A)^A_\alpha &= -\left(\mathcal{C}^\delta_{\alpha\beta}\, y^C_\delta \frac{\partial H}{\partial y^C_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right) \end{aligned} \tag{4.41}$$

Next, let $\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^*,\ \psi(\mathbf{t}) = (\psi^i(\mathbf{t}), \psi^A_\alpha(\mathbf{t}))$ be an integral section of ξ, that is, ψ is an integral section of $(\rho^{\widetilde{\tau}^*}(\xi), \ldots, \rho^{\widetilde{\tau}^*}(\xi))$ the k-vector field on $\overset{k}{\oplus} E^*$ associated to ξ. Thus the following expressions holds:

$$\xi^\beta_A \rho^i_\beta = \frac{\partial \psi^i}{\partial t^A} \ , \ (\xi_A)^B_\beta = \frac{\partial \psi^B_\beta}{\partial t^A} \ . \tag{4.42}$$

Finally, from (4.41) and (4.42) we deduce that ψ satisfies the following system of partial differential equations.

$$\frac{\partial \psi^i}{\partial t^A} = \frac{\partial H}{\partial y^A_\alpha} \rho^i_\alpha \quad \text{and} \quad \sum_{A=1}^k \frac{\partial \psi^A_\alpha}{\partial t^A} = -\left(\mathcal{C}^\delta_{\alpha\beta}\, \psi^A_\delta \frac{\partial H}{\partial y^A_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right) .$$

$\blacksquare$

Remark 4.26.

(i) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$ the equation (4.38) is the geometric version of the Hamilton field equation in the standard k-symplectic formalism. This fact will be explain after.

(ii) In the particular case $k = 1$, this theorem summarized the Hamiltonian Mechanics on Lie algebroids, see section 3.2 in [8] or section 3.3 on [19].

$\diamond$

As a final remark in this subsection, it is interesting to point out that the standard Hamiltonian k-symplectic formalism is a particular case of the Hamiltonian formalism on Lie algebroids. In this case $E = TQ$ and $\rho_E = id_{TQ}$ as we have comment in the point (i) of the remark 4.26. We have:

- The manifold $\overset{k}{\oplus} E^*$ is identified with $(T^1_k)^*Q$; $\mathcal{T}^{TQ}((T^1_k)^*Q)$ with $T((T^1_k)^*Q)$; and $(\mathcal{T}^{TQ})^1_k((T^1_k)^*Q)$ with $T^1_k((T^1_k)^*Q)$.

- A section
$$\xi : \overset{k}{\oplus} E^* \to (\mathfrak{T}^E)^1_k(\overset{k}{\oplus} E^*)$$
corresponds to a k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on $(T^1_k)^*Q$, that is, ξ is a section of $\tau^k_{(T^1_k)^*Q} : T^1_k((T^1_k)^*Q) \to (T^1_k)^*Q$.
- Let f be a function defined on $(T^1_k)^*Q$ then
$$(d^{\mathfrak{T}^E(\overset{k}{\oplus} E^*)} f)(Y) = df(Y)$$
where df denotes the usual differential and Y is a vector field on $(T^1_k)^*Q$.
- It is satisfies that
$$\Omega^A(X,Y) = \omega^A(X,Y) \quad (A = 1, \dots, k)$$
where $\omega^A,\ A = 1, \dots, k$ are the canonical k-symplectic 2-forms on $(T^1_k)^*Q$.
- Thus, in the standard Hamiltonian k-symplectic formalism the equation (4.38) writes as follow:
$$\sum_{A=1}^k \imath_{\xi_A}\omega^A = dH\,.$$

As consequence of the Theorem 4.24 and the five above remarks, we reobtain the standard Hamiltonian k-symplectic formalism, which can be summarized in the following

Corollary 4.27. *Let $H : (T^1_k)^*Q \to \mathbb{R}$ be a Hamiltonian formalism and $\xi = (\xi_1, \dots, \xi_k)$ be a k-vector field on $(T^1_k)^*Q$ such that*
$$\sum_{A=1}^k \imath_{\xi_A}\omega^A = dH\,.$$

If $\psi : \mathbb{R}^k \to (T^1_k)^*Q,\ \psi(\mathbf{t}) = (\psi^i(\mathbf{t}), \psi^A_i(\mathbf{t}))$ is an integral section of ξ, then is a solution to the Hamilton field equation in the standard k-symplectic formalism, that is,

$$\sum_{A=1}^k \frac{\partial \psi^A_i}{\partial t^A}\Big|_{\mathbf{t}} = -\frac{\partial H}{\partial q^i}\Big|_{\psi(\mathbf{t})}, \quad \frac{\partial \psi^i}{\partial t^A}\Big|_{\mathbf{t}} = \frac{\partial H}{\partial p^A_i}\Big|_{\psi(\mathbf{t})}, \quad i = 1\dots, n\,. \tag{4.43}$$

4.3. **The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian k-symplectic formalism on Lie algebroids.** In this section we introduce the Legendre transformation on the k-symplectic framework on Lie algebroids and we establish the equivalence between the Lagrangian and Hamiltonian formulation when we consider a hyperregular Lagrangian, This fact extends the analogous results of the standard case.

Let $L : \overset{k}{\oplus} E \to \mathbb{R}$ be a Lagrangian function and $\Theta^A_L \colon \overset{k}{\oplus} E \to [\mathfrak{T}^E(\overset{k}{\oplus} E)]^*$, $(A = 1, \dots, k)$ be the Poincaré-Cartan 1-sections associated with L, which was defined in (4.20).

Definition 4.28. *We introduce the Legendre transformation associated with L as the smooth map*
$$\mathfrak{Leg} : \overset{k}{\oplus} E \to \overset{k}{\oplus} E^*$$
defined by
$$\mathfrak{Leg}(b_{1_q}, \dots, b_{k_q}) = \Big([\mathfrak{Leg}(b_{1_q}, \dots, b_{k_q})]^1, \dots, [\mathfrak{Leg}(b_{1_q}, \dots, b_{k_q})]^k\Big)$$
where
$$[\mathfrak{Leg}(b_{1_q}, \dots, b_{k_q})]^A(e_q) = \frac{d}{ds} L(b_{1_q}, \dots, b_{Aq} + s e_q, \dots, b_{k_q})\Big|_{s=0},$$
being $e_q \in E_q$.

En other words, for each A we can write

$$[\mathfrak{Leg}(b_{1_q},\dots,b_{k_q})]^A(e_q)=\Theta_L^A(b_{1_q},\dots,b_{k_q})(Z)\,, \tag{4.44}$$

where Z is a point in the fiber of $(\mathcal{T}^E(\overset{k}{\oplus} E))_{\mathbf{b}_q}$, over the point

$$\mathbf{b}_q=(b_{1_q},\dots,b_{k_q})\in\overset{k}{\oplus} E$$

such that

$$\widetilde{\tau}_1(Z)=e_q$$

being

$$\widetilde{\tau}_1:\mathcal{T}^E(\overset{k}{\oplus} E)=E\times_{TQ}T(\overset{k}{\oplus} E)\to E$$

is the projection over the first factor. Therefore Z is of the form $Z=(e_q,v_{b_q})$.

The map $\mathfrak{Leg}$ is well-defined and its local expression is

$$\mathfrak{Leg}(q^i,y^\alpha_A)=(q^i,\frac{\partial L}{\partial y^\alpha_A})\,.$$

From this local expression it is easy to prove that the Lagrangian L is regular if an only if $\mathfrak{Leg}$ is a local diffeomorphism.

Remark 4.29. When $E=TQ$ the Legendre transformation defined here coincides with the Legendre transformation introduced by Günther in [14]. $\diamond$

The Legendre transformation, $\mathfrak{Leg}$, induce a map

$$\mathcal{T}^E\mathfrak{Leg}:\mathcal{T}^E(\overset{k}{\oplus} E)\equiv E\times_{TQ}T(\overset{k}{\oplus} E)\to\mathcal{T}^E(\overset{k}{\oplus} E^*)\equiv E\times_{TQ}T(\overset{k}{\oplus} E^*)$$

defined as follow

$$\mathcal{T}^E\mathfrak{Leg}(e_q,v_{\mathbf{b}_q})=\Big(e_q,(\mathfrak{Leg})_*(\mathbf{b}_q)(v_{\mathbf{b}_q})\Big)\,,$$

where $e_q\in E_q$, $\mathbf{b}_q\in\overset{k}{\oplus} E$ y $(e_q,v_{\mathbf{b}_q})\in\mathcal{T}^E(\overset{k}{\oplus} E)\equiv E\times_{TQ}T(\overset{k}{\oplus} E)$. Notice that the following diagram is commutative

$$\begin{tikzcd}
\overset{k}{\oplus} E \arrow[rr, "\mathfrak{Leg}"] \arrow[dr, "\widetilde{\tau}"'] & & \overset{k}{\oplus} E^* \arrow[dl, "\widetilde{\tau}^*"] \\
& Q &
\end{tikzcd}$$

and thus $\mathcal{T}^E\mathfrak{Leg}$ is well-defined.

If we consider local coordinates on $\mathcal{T}^E(\overset{k}{\oplus} E)$ (resp. $\mathcal{T}^E(\overset{k}{\oplus} E^*)$), see (4.2) and (4.30), the local expression of $\mathcal{T}\mathfrak{Leg}$ is

$$\mathcal{T}^E\mathfrak{Leg}(q^i,y^\alpha_A,z^\alpha,w^\beta_B)=(q^i,\frac{\partial L}{\partial y^\alpha_A},z^\alpha,z^\alpha\rho^i_\alpha\frac{\partial^2 L}{\partial q^i\partial y^\gamma_C}+w^\beta_B\frac{\partial^2 L}{\partial y^\gamma_C\partial y^\beta_B})\,. \tag{4.45}$$

Theorem 4.30. *The pair $(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})$ is a morphism between the Lie algebroids $(\mathcal{T}^E(\overset{k}{\oplus} E),\rho^{\widetilde{\tau}},[\![\cdot,\cdot]\!]^{\widetilde{\tau}})$ and $(\mathcal{T}^E(\overset{k}{\oplus} E^*),\rho^{\widetilde{\tau}^*},[\![\cdot,\cdot]\!]^{\widetilde{\tau}^*})$. Moreover, if Θ^A_L and Ω^A_L (respectively, Θ^A and Ω^A) are the Poincaré-Cartan 1-sections and 2-sections associated with $L\colon\ \overset{k}{\oplus} E\to\mathbb{R}$ (respectively, the Liouville 1-sections and 2-sections on $\mathcal{T}^E(\overset{k}{\oplus} E^*)$), then*

$$(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\Theta^A=\Theta^A_L,\qquad(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\Omega^A=\Omega^A_L\,,\quad 1\leq A\leq k\,. \tag{4.46}$$

(*Proof*) Firstly we have to prove that $(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})$ is a Lie algebroid morphism.

$$\begin{array}{ccc}
\mathcal{T}^E(\overset{k}{\oplus} E) & \xrightarrow{\mathcal{T}^E\mathfrak{Leg}} & \mathcal{T}^E(\overset{k}{\oplus} E^*) \\
{\scriptstyle \widetilde{\tau}_{\overset{k}{\oplus}E}}\downarrow & & \downarrow{\scriptstyle \widetilde{\tau}_{\overset{k}{\oplus}E^*}} \\
\overset{k}{\oplus} E & \xrightarrow{\mathfrak{Leg}} & \overset{k}{\oplus} E^*
\end{array}$$

Suppose that (q^i) are local coordinates on Q, that $\{e_\alpha\}$ is a local basis of $\mathrm{Sec}(E)$ and denote by $\{\mathcal{X}_\alpha,\mathcal{V}^A_\alpha\}$ (respectively, $\{\mathcal{Y}_\alpha,\mathcal{U}^\alpha_A\}$) the corresponding local basis of sections of $\widetilde{\tau}_{\overset{k}{\oplus}E}\colon \mathcal{T}^E(\overset{k}{\oplus} E)\to\overset{k}{\oplus} E$ (respectively, $\widetilde{\tau}_{\overset{k}{\oplus}E^*}\colon \mathcal{T}^E(\overset{k}{\oplus} E^*)\to\overset{k}{\oplus} E^*$).

Then, using (3.5), (4.6) and (4.45), by a straightforward computation we deduce that

$$(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*(\mathcal{Y}^\alpha)=\mathcal{X}^\alpha \quad , \quad (\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*(\mathcal{U}_\alpha^A)=d^{\mathcal{T}^E(\overset{k}{\oplus}E)}\left(\frac{\partial L}{\partial y^\alpha_A}\right), \tag{4.47}$$

for each $\alpha=1,\dots,m$ and $A=1,\dots,k$ where $\{\mathcal{X}^\alpha,\mathcal{V}_A^\alpha\}$ and $\{\mathcal{Y}^\alpha,\mathcal{U}^A_\alpha\}$ denotes the dual basis of $\{\mathcal{X}_\alpha,\mathcal{V}^A_\alpha\}$ and $\{\mathcal{Y}_\alpha,\mathcal{U}^\alpha_A\}$ respectively.

Thus, taking into account this identities, from (4.6) and (4.34) we conclude that

$$\begin{aligned}
(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*(d^{\mathcal{T}^E(\overset{k}{\oplus}E^*)}f) &= d^{\mathcal{T}^E(\overset{k}{\oplus}E)}(f\circ\mathfrak{Leg})\\
(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*(d^{\mathcal{T}^E(\overset{k}{\oplus}E^*)}\mathcal{Y}^\alpha) &= d^{\mathcal{T}^E(\overset{k}{\oplus}E)}((\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\mathcal{Y}^\alpha)\\
(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*(d^{\mathcal{T}^E(\overset{k}{\oplus}E^*)}\mathcal{U}^A_\alpha) &= d^{\mathcal{T}^E(\overset{k}{\oplus}E)}((\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\mathcal{U}^A_\alpha)\,,
\end{aligned}$$

for all function $f\in\mathcal{C}^\infty(\overset{k}{\oplus} E^*)$ and for all α and A.

Consequently, the pair $(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})$ is a Lie algebroid morphism

Next we will check that $(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\Theta^A=\Theta^A_L$ holds.

From (3.5), (4.35) and (4.44) we obtain:

$$\begin{aligned}
[(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\Theta^A]_{\mathbf{b}_q}(e_q,v_{\mathbf{b}_q}) &= \Theta^A_{\mathfrak{Leg}(\mathbf{b}_q)}(e_q,(\mathfrak{Leg})_*(\mathbf{b}_q)(v_{\mathbf{b}_q}))\\
&= [\mathfrak{Leg}(\mathbf{b}_q)]^A(e_q)=\Theta^A_L(\mathbf{b}_q)(e_q,v_{\mathbf{b}_q})\,.
\end{aligned}$$

Finally, since $(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})$ is a Lie algebroid morphism and taking into account the last identity we deduce that:

$$(\mathcal{T}^E\mathfrak{Leg},\mathfrak{Leg})^*\Omega^A=\Omega^A_L\,.$$

■

Remark 4.31. In the particular case $k=1$ this theorem corresponds with the Theorem 3.12 of [19].

In the case $E=TQ$ and $\rho_{TQ}=id_{TQ}$ this theorem establishes the relation between the Lagrangian and Hamiltonian forms in the standard k-symplectic approach. ◇

Next, we will assume that L is *hyperregular*, that is, $\mathfrak{Leg}$ is a global diffeomorphism. In this case we may consider the Hamiltonian function $H\colon \overset{k}{\oplus} E^*\to\mathbb{R}$ defined by

$$H=E_L\circ(\mathfrak{Leg})^{-1},$$

where E_L es the Lagrangian energy associated with L given by (4.24). Here $(\mathfrak{Leg})^{-1}$ is the inverse of the Legendre transformation

$$\begin{tikzcd} \overset{k}{\oplus} E^* \arrow[r, "\mathfrak{Leg}^{-1}"] \arrow[dr, dashed, "H"'] & \overset{k}{\oplus} E \arrow[d, "E_L"] \\ & \mathbb{R} \end{tikzcd}$$

Lemma 4.32. *If the Lagrangian L is hyperregular then $\mathcal{T}^E\mathfrak{Leg}$ is a diffeomorphism*

(*Proof*) The condition L hyperregular means that $\mathfrak{Leg}$ is a global diffeomorphism, that is, there exists its inverse map

$$\mathfrak{Leg}^{-1}\colon\ \overset{k}{\oplus} E^* \to \overset{k}{\oplus} E.$$

We define the inverse map to $\mathcal{T}^E\mathfrak{Leg}$ as the mapping

$$(\mathcal{T}^E\mathfrak{Leg})^{-1} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to \mathcal{T}^E(\overset{k}{\oplus} E)$$

given by

$$(\mathcal{T}^E\mathfrak{Leg})^{-1}(a_q, v_{\mathbf{b}_q^*}) = \Big(a_q, (\mathfrak{Leg}^{-1})_*(\mathbf{b}_q^*)(v_{\mathbf{b}_q^*})\Big)\,,$$

where $a_q \in E$, $\mathbf{b}_q^* \in \overset{k}{\oplus} E^*$ and $(a_q, v_{\mathbf{b}_q^*}) \in \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*)$.

Therefore, $\mathcal{T}^E\mathfrak{Leg}$ is a diffeomorphism. ∎

The following theorem establishes the equivalence between the Lagrangian and Hamiltonian k-symplectic formulation on Lie algebroids.

Theorem 4.33. *Let L be a hyperregular Lagrangian. There is a bijective correspondence between the set of maps $\eta : \mathbb{R}^k \to \overset{k}{\oplus} E$ such that η is an integral section of a solution ξ_L of the geometric Euler-Lagrange equations (4.26) and the set of maps $\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^*$ which are integral sections of some solution ξ_H of the geometric Hamilton equations (4.38).*

(*Proof*)

The proof is similar to the standard case, see [46]. An outline of the proof is the following:

Let $\xi_L = (\xi_L^1, \dots, \xi_L^k) : \overset{k}{\oplus} E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E)$ be a solution of the geometric Euler-Lagrange equations on Lie algebroids (4.26), then $\xi_H = (\xi_H^1, \dots, \xi_H^k)$ where each

$$\xi_H^A = \mathcal{T}^E\mathfrak{Leg} \circ \xi_L^A \circ (\mathfrak{Leg})^{-1}$$

is a solution of (4.38).

Moreover, *if $\eta : \mathbb{R}^k \to \overset{k}{\oplus} E$ is an integral section of $\xi_L = (\xi_L^1, \dots, \xi_L^k)$, then*

$$\mathfrak{Leg} \circ \eta : \mathbb{R}^k \to \overset{k}{\oplus} E^*$$

is an integral section of $\xi_H = (\xi_H^1, \dots, \xi_H^k)$ being

$$\xi_H^A = \mathcal{T}^E\mathfrak{Leg} \circ \xi_L^A \circ (\mathfrak{Leg})^{-1}.$$

The converse is proved in a similar way. ∎

Remark 4.34. If we rewrite the results of this subsection in the particular case $k = 1$ we obtain the equivalence between the Lagrangian an Hamiltonian Autonomous Mechanics on Lie algebroids, see for instance [8].

When $E = TQ$ and $\rho_{TQ} = id_{TQ}$, we obtain the equivalence between the Lagrangian and Hamiltonian formulation in the standard k-symplectic framework, see [46] $\diamond$

5. Examples

Harmonic mappings. ([6, 5, 44, 48]) Here, we consider harmonic mappings $\phi : \mathbb{R}^2 \to G$ with values in an arbitrary Lie group G with bi-invariant metric $\langle \cdot, \cdot \rangle$. In the continuous case (see [44]), the harmonic mapping Lagrangian is given by

$$L(\phi, \phi_x, \phi_y) = \frac{1}{2}\langle \phi^{-1}\phi_x, \phi^{-1}\phi_x\rangle + \frac{1}{2}\langle \phi^{-1}\phi_y, \phi^{-1}\phi_y\rangle \,, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle$ is the *Killing form* on $\mathfrak{g}$ and ϕ_x, ϕ_y denotes the partial derivatives of ϕ respect to the local coordinates (x, y) of $\mathbb{R}^2$. The associated field equations are $\tau(\phi) = 0$ where $\tau(\phi)$ is the tension of ϕ, defined as

$$\tau(\phi)^i = h^{AB}\left(\frac{\partial^2\phi^i}{\partial t^A \partial t^B} - \Gamma^C_{AB}\frac{\partial \phi^i}{\partial t^C} + \mathcal{C}^i_{jk}\frac{\partial \phi^j}{\partial t^A}\frac{\partial \phi^k}{\partial t^B}\right) \,, \quad i = 1, \dots, \dim G \,,$$

where h_{AB} are the components of a metric on $\mathbb{R}^2$, with Christoffel symbols Γ^C_{AB} and $\mathcal{C}^i_{jk}$ are the Christoffel symbols of the bi-invariant metric on G. In our case, h_{AB} is of course just the flat Euclidian metric.

We will only treat the case of the harmonic maps that take values in $G = SO(3)$, embedded in $\mathfrak{gl}(3)$, in which case the *Killing form* $\langle \cdot, \cdot \rangle$ is just the trace

$$\langle \xi, \eta \rangle = -\, trace(\xi\eta) \,.$$

Let us observe that in this case, the Lagrangian (5.1) is represented as a function $L : TSO(3) \oplus TSO(3) \to \mathbb{R}$ defined on $T^1_2(SO(3))$.

Taking into account that $T^1_2(SO(3)) \cong SO(3)\times\mathfrak{so}(3)\times\mathfrak{so}(3)$, we make the identifications

$$T^1_2(SO(3))/SO(3) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$$

and we consider the projection l of L to $\mathfrak{so}(3) \times \mathfrak{so}(3)$ given by

$$l(\xi_1, \xi_2) = -\frac{1}{2}trace(\xi_1^2) - \frac{1}{2}trace(\xi_2^2) \,, \quad \xi_1, \xi_2 \in \mathfrak{so}(3) \,.$$

Let $\{E_1, E_2, E_3\}$ be a basis of $\mathfrak{so}(3)$, then $\xi_i = y_i^\alpha E_\alpha$, $i = 1, 2$ and thus, l is locally given by

$$l(y_1^\alpha, y_2^\alpha) = \sum_{\alpha=1}^{3}\left((y_1^\alpha)^2 + (y_2^\alpha)^2\right) .$$

Since a Lie algebra is a example of a Lie algebroid we can apply the theory developed in Section 4.1 and thus the Euler-Lagrange (4.28) equations are given, in this case, by

$$\left.\begin{array}{rcl} \dfrac{\partial y_1^\alpha}{\partial t^1} + \dfrac{\partial y_2^\alpha}{\partial t^2} & = & 0 \\[2ex] \dfrac{\partial y_A^\alpha}{\partial t^B} - \dfrac{\partial y_B^\alpha}{\partial t^A} + \mathcal{C}^\alpha_{\beta\gamma} y_B^\beta y_A^\gamma & = & 0 \end{array}\right. \quad (\alpha = 1, 2, 3;\ A = 1, 2)$$

Poisson sigma model. Consider a Poisson manifold (Q, Λ). Then the cotangent bundle T^*Q has a Lie algebroid structure, where the anchor is

$$\begin{array}{rcl} \rho\colon T^*Q & \to & TQ \\ \beta & \mapsto & \Lambda(\beta, \cdot) \end{array}$$

and the bracket is

$$[\alpha, \beta] = \imath_{\rho(\alpha)} d\beta - \imath_{\rho(\beta)} d\alpha - d\Lambda(\alpha, \beta)\,.$$

In local coordinates, the bivector Λ has the local expression

$$\Lambda = \frac{1}{2}\Lambda^{ij}\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}\,.$$

We can consider the Lagrangian for the sigma Poisson model as a function defined on $T^*Q \oplus T^*Q$. Thus if (q^i, p^i_1, p^i_2) denotes the local coordinates on $T^*Q \oplus T^*Q$, the local expression of the Lagrangian is (see [32])

$$L = -\frac{1}{2}\Lambda^{ij} p^1_i p^2_j\,.$$

A long but straightforward calculation shows that the Euler-Lagrange equation (4.28) are in this case

$$\begin{array}{l} \dfrac{1}{2}\Lambda^{ij}\left(\dfrac{\partial p^2_i}{\partial t^1} - \dfrac{\partial p^1_i}{\partial t^2} + \dfrac{\partial \Lambda^{kl}}{\partial q^i} p^1_k p^2_l\right) = 0\,, \\ \dfrac{\partial q^i}{\partial t^A} + \Lambda^{ij} p^A_j = 0\ , \\ \dfrac{\partial p^2_i}{\partial t^1} - \dfrac{\partial p^1_i}{\partial t^2} + \dfrac{\partial \Lambda^{kl}}{\partial q^i} p^1_k p^2_l = 0\ , \end{array}$$

In view of the morphism condition, we see that the first equation vanishes. Thus the field equations are just

$$\begin{array}{l} \dfrac{\partial q^i}{\partial t^A} + \Lambda^{ij} p^A_j = 0\ , \\ \dfrac{\partial p^2_i}{\partial t^1} - \dfrac{\partial p^1_i}{\partial t^2} + \dfrac{\partial \Lambda^{kl}}{\partial q^i} p^1_k p^2_l = 0\ , \end{array}$$

where a solution is a field $\phi : \mathbb{R}^2 \to T^*Q \oplus T^*Q$, locally given by

$$\phi(\mathbf{t}) = (q^i(\mathbf{t}), p^1_i(\mathbf{t}), p^2_i(\mathbf{t})).$$

Consider the 1-forms on $\mathbb{R}^2$ given by $P_j = p^1_j dt^1 + p^2_j dt^2$ $(j = 1, \dots, n)$, then the above equations can be written as

$$\begin{aligned} &d\phi^j + \Lambda^{jk} P_k = 0 \\ &dP_j + \frac{1}{2}\Lambda^{kl}_{,j} P_k \wedge P_L = 0\,, \end{aligned}$$

that is the conventional form of the field equations for the Poisson-sigma model [43]

Remark 5.1. Poisson sigma models were originally introduced by Schaller, Strobl, [42], and Ikeda [16] so as to unify several two-dimensional models of gravity and to cast them into a common form with Yang-Mills theories. ◇

Systems with symmetry. We consider a principal bundle $\pi : \bar{Q} \longrightarrow Q = \bar{Q}/G$. Let $A : T\bar{Q} \longrightarrow \mathfrak{g}$ be fixed principal connection with curvature $B : T\bar{Q} \oplus T\bar{Q} \longrightarrow \mathfrak{g}$. The connection A determines an isomorphism between the vector bundles $T\bar{Q}/G \to Q$ and $TQ \oplus \widetilde{\mathfrak{g}} \longrightarrow Q$ where $\widetilde{\mathfrak{g}} = (\bar{Q} \times \mathfrak{g})/G$ is the adjoint bundle (see [7]):

$$[v_{\bar{q}}] \leftrightarrow T_{\bar{q}}\pi(v_{\bar{q}}) \oplus [(\bar{q}, A(v_{\bar{q}}))]$$

where $v_{\bar{q}} \in T_{\bar{q}}\bar{Q}$. The connection permits us to obtain a local basis of sections of $\mathrm{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \mathrm{Sec}(\widetilde{\mathfrak{g}})$ as follows. Let $\mathfrak{e}$ be the identity element of the Lie group G and assume that there are local coordinates (q^i), $1 \leq i \leq \dim Q$ and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$. The corresponding sections of the adjoint bundle are the left invariant vector fields ξ_a^L:

$$\xi_a^L(g) = T_{\mathfrak{e}}L_g(\xi_a)$$

where $L_g : G \longrightarrow G$ is the left translation by $g \in G$. If

$$A\left(\frac{\partial}{\partial q^i}_{(q,\mathfrak{e})}\right) = A_i^a \xi_a$$

then corresponding horizontal lift on the trivialization $U \times G$ are the vector fields

$$\left(\frac{\partial}{\partial q^i}\right)^h = \frac{\partial}{\partial q^i} - A_i^a \xi_a^L$$

The set

$$\left\{\left(\frac{\partial}{\partial q^i}\right)^h, \xi_a^L\right\}$$

are by construction G-invariant and therefore, they constitute a local basis of sections $\{e_i, e_a\}$ of $\mathrm{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \mathrm{Sec}(\widetilde{\mathfrak{g}})$. Denote by (q^i, y^i, y^a) the induced local coordinates of $T\bar{Q}/G$. If $\mathfrak{C}_{ab}^c$ are the structure constants of the Lie algebra

$$B\left(\frac{\partial}{\partial q^i}_{(q,\mathfrak{e})}, \frac{\partial}{\partial q^j}_{(q,\mathfrak{e})}\right) = B_{ij}^a \xi_a$$

where

$$B_{ij}^c = \frac{\partial A_i^c}{\partial q^j} - \frac{\partial A_j^c}{\partial q^i} - \mathfrak{C}_{ab}^c A_i^a A_j^b \,.$$

then the structure functions of the Lie algebroid $T\bar{Q}/G \to Q$ are determined by the following relations (see [19]):

$$\begin{aligned}
[\![e_i, e_j]\!]_{T\bar{Q}/G} &= -B_{ij}^c e_c \\
[\![e_i, e_a]\!]_{T\bar{Q}/G} &= \mathfrak{C}_{ab}^c A_i^b e_c \\
[\![e_a, e_b]\!]_{T\bar{Q}/G} &= \mathfrak{C}_{ab}^c e_c \\
\rho_{T\bar{Q}/G}(e_i) &= \frac{\partial}{\partial q^i} \\
\rho_{T\bar{Q}/G}(e_a) &= 0\,.
\end{aligned}$$

Now, consider a Lagrangian function $L : \overset{k}{\oplus} T\bar{Q}/G \longrightarrow \mathbb{R}$ then the Euler-lagrange field equations are:

$$\begin{aligned} \frac{d}{dt^A}\left(\frac{\partial L}{\partial y^i_A}\right) &= \frac{\partial L}{\partial q^i} + B^c_{ij} y^j_C \frac{\partial L}{\partial y^c_C} - \mathfrak{C}^c_{ab} A^b_i y^a_C \frac{\partial L}{\partial y^c_C} \\ \frac{d}{dt^A}\left(\frac{\partial L}{\partial y^a_A}\right) &= \mathfrak{C}^c_{ab} A^b_i y^i_C \frac{\partial L}{\partial y^c_C} - \mathfrak{C}^c_{ab} y^b_C \frac{\partial L}{\partial y^c_C} \\ 0 &= \frac{\partial y^i_A}{\partial t^B} - \frac{\partial y^i_B}{\partial t^A} \\ 0 &= \frac{\partial y^c_A}{\partial t^B} - \frac{\partial y^c_B}{\partial t^A} - B^c_{ij} y^i_B y^j_A + \mathfrak{C}^c_{ab} A^b_i y^i_B y^a_A + \mathfrak{C}^c_{ab} y^b_A y^a_B \end{aligned}$$

In the case when Q is a single point, that is $\bar{Q} = G$ then $T\bar{Q}/G = \mathfrak{g}$ and then the Lagrangian is defined as a function $L : \overset{k}{\oplus} \mathfrak{g} \longrightarrow \mathbb{R}$ and then the previous equations are reduced now to

$$\begin{aligned} \frac{d}{dt^A}\left(\frac{\partial L}{\partial y^a_A}\right) &= -c^c_{ab} y^b_C \frac{\partial L}{\partial y^c_C} \\ 0 &= \frac{\partial y^c_A}{\partial t^B} - \frac{\partial y^c_B}{\partial t^A} + \mathfrak{C}^c_{ab} y^b_A y^a_B \end{aligned}$$

which are a local expression of Euler-Poincaré equations, see for instance [6] and [32].

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